Chapter 6

Lucy: "I've just come up with the perfect theory. It's my theory that Beethoven would have written even better music if he had been married."

Schroeder: "What's so perfect about that theory?"

Lucy: "It can't be proved one way or the other!"

—Charles Schulz, *Peanuts*, 1976.

Copeland, Weston + Shastri
Financial Theory + Corporate Policy 4th edn

Market Equilibrium: CAPM and APT

A. Introduction

THIS CHAPTER EXTENDS THE CONCEPT of market equilibrium to determine the market price for risk and the appropriate measure of risk for a single asset. One economic model to solve this problem, called the *capital asset pricing model* (CAPM), was developed almost simultaneously by Sharpe [1963, 1964] and Treynor [1961], and then further developed by Mossin [1966], Lintner [1965, 1969], and Black [1972]. It shows that the equilibrium rates of return on all risky assets are a function of their covariance with the market portfolio. A second important equilibrium pricing model, called the *arbitrage pricing theory* (APT), was developed by Ross [1976]. It is similar to the CAPM in that it is also an equilibrium asset pricing model. The return on any risky asset is seen to be a linear combination of various common factors that affect asset returns. It is more general than the CAPM because it allows numerous factors to explain the equilibrium return on a risky asset. However, it is in the same spirit as the CAPM. In fact, the CAPM can be shown to be a special case of the APT.

This chapter first develops the CAPM and its extensions, and then summarizes the empirical evidence relating to its validity. Thereafter, the APT will be developed and the empirical evidence for it will be described. We begin with a list of the assumptions that were first used to derive the CAPM.

The CAPM is developed in a hypothetical world where the following assumptions are made about investors and the opportunity set:

1. Investors are risk-averse individuals who maximize the expected utility of their wealth.
2. Investors are price takers and have homogeneous expectations about asset returns that have a joint normal distribution.

3. There exists a risk-free asset such that investors may borrow or lend unlimited amounts at a risk-free rate.
4. The quantities of assets are fixed. Also, all assets are marketable and perfectly divisible.
5. Asset markets are frictionless, and information is costless and simultaneously available to all investors.
6. There are no market imperfections such as taxes, regulations, or restrictions on short selling.

Many of these assumptions have been discussed earlier. However, it is worthwhile to discuss some of their implications. For example, if markets are frictionless, the borrowing rate equals the lending rate, and we are able to develop a linear efficient set called the capital market line (Fig. 5.17 and Eq. 5.34). If all assets are divisible and marketable, we exclude the possibility of human capital as we usually think of it. In other words, slavery is allowed in the model. We are all able to sell (not rent for wages) various portions of our human capital (e.g., typing ability or reading ability) to other investors at market prices. Another important assumption is that investors have homogeneous beliefs. They all make decisions based on an identical opportunity set. In other words, no one can be fooled because everyone has the same information at the same time. Also, since all investors maximize the expected utility of their end-of-period wealth, the model is implicitly a one-period model.

Although not all these assumptions conform to reality, they are simplifications that permit the development of the CAPM, which is extremely useful for financial decision making because it quantifies and prices risk. Most of the restrictive assumptions will be relaxed later on.

B. The Efficiency of the Market Portfolio

Proof of the CAPM requires that in equilibrium the market portfolio must be an efficient portfolio. It must lie on the upper half of the minimum variance opportunity set graphed in Fig. 6.1. One way to establish its efficiency is to argue that because investors have homogeneous expectations, they will all perceive the same minimum variance opportunity set.[1] Even without a risk-free asset, they will all select efficient portfolios regardless of their individual risk tolerances. As shown in Fig. 6.1, individual I chooses efficient portfolio B, whereas individual II, who is less risk averse, chooses efficient portfolio C. Given that all individuals hold positive proportions of their wealth in efficient portfolios, then the market portfolio must be efficient because (1) the market is simply the sum of all individual holdings and (2) all individual holdings are efficient.

Thus, in theory, when all individuals have homogeneous expectations, the market portfolio must be efficient. Without homogeneous expectations, the market portfolio is not necessarily efficient, and the equilibrium model of capital markets that is derived in the next section does not necessarily hold. Thus the efficiency of the market portfolio and the capital asset pricing model are inseparable, joint hypotheses. It is not possible to test the validity of one without the other. We shall return to this important point when we discuss Roll's critique later in the chapter.

[1] For a more rigorous proof of the efficiency of the market portfolio, see Fama [1976, Chapter 8].

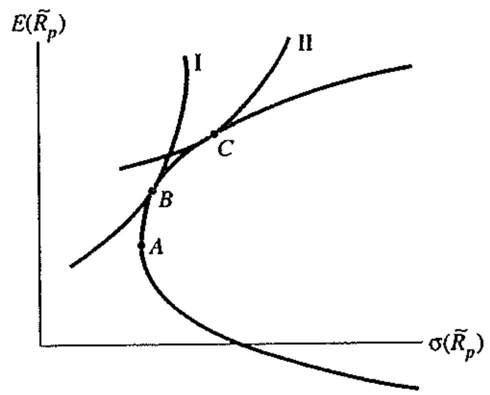


Figure 6.1 All investors select efficient portfolios.

C. Derivation of the CAPM

Figure 6.2 shows the expected return and standard deviation of the market portfolio, M, the risk-free asset, R_f, and a risky asset, I. The straight line connecting the risk-free asset and the market portfolio is the *capital market line* (for example, see Fig. 5.17, in Chapter 5). We know that if a market equilibrium is to exist, the prices of all assets must adjust until all are held by investors. There can be no excess demand. In other words, prices must be established so that the supply of all assets equals the demand for holding them. Consequently, in equilibrium the market portfolio will consist of all marketable assets held in proportion to their value weights. The equilibrium proportion of each asset in the market portfolio must be

$$w_i = \frac{\text{market value of the individual asset}}{\text{market value of all assets}}. \tag{6.1}$$

A portfolio consisting of $a\%$ invested in risky asset I and $(1-a)\%$ in the market portfolio will have the following mean and standard deviation:

$$E(\tilde{R}_p) = aE(\tilde{R}_i) + (1-a)E(\tilde{R}_m), \tag{6.2}$$

$$\sigma(\tilde{R}_p) = \left[a^2\sigma_i^2 + (1-a)^2\sigma_m^2 + 2a(1-a)\sigma_{im}\right]^{1/2}, \tag{6.3}$$

where

σ_i^2 = the variance of risky asset I,
σ_m^2 = the variance of the market portfolio,
σ_{im} = the covariance between asset I and the market portfolio.

We shall see shortly that the market portfolio already contains asset I held according to its market value weight. In fact the definition of the market portfolio is that it consists of all assets held according to their market value weights. The opportunity set provided by various combinations of the risky asset and the market portfolio is the line IMI' in Fig. 6.2. The change in the mean

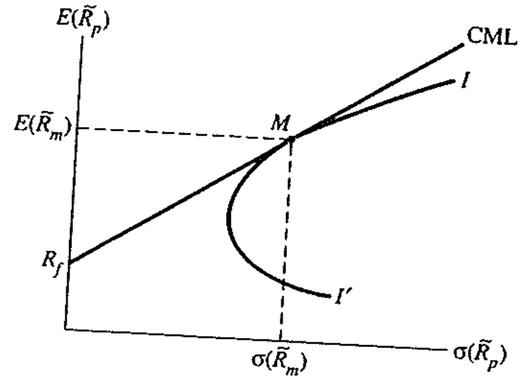


Figure 6.2 The opportunity set provided by combinations of risky asset I and the market portfolio, M.

and standard deviation with respect to the percentage of the portfolio, a, invested in asset I is determined as follows:

$$\frac{\partial E(\widetilde{R}_p)}{\partial a} = E(\widetilde{R}_i) - E(\widetilde{R}_m), \tag{6.4}$$

$$\frac{\partial \sigma(\widetilde{R}_p)}{\partial a} = \frac{1}{2}\left[a^2\sigma_i^2 + (1-a)^2\sigma_m^2 + 2a(1-a)\sigma_{im}\right]^{-1/2} \times \left[2a\sigma_i^2 - 2\sigma_m^2 + 2a\sigma_m^2 + 2\sigma_{im} - 4a\sigma_{im}\right]. \tag{6.5}$$

Sharpe's and Treynor's insight, which allowed them to use the above facts to determine a market equilibrium price for risk, was that in equilibrium the market portfolio already has the value weight, w_i percent, invested in the risky asset I. Therefore the percentage a in the above equations is the excess demand for an individual risky asset. But we know that in equilibrium the excess demand for any asset must be zero. Prices will adjust until all assets are held by someone. Therefore if Eqs. (6.4) and (6.5) are evaluated where excess demand, a, equals zero, then we can determine the equilibrium price relationships at point M in Fig. 6.2. This will provide the equilibrium price of risk. Evaluating Eqs. (6.4) and (6.5) where $a = 0$, we obtain

$$\left.\frac{\partial E(\widetilde{R}_p)}{\partial a}\right|_{a=0} = E(\widetilde{R}_i) - E(\widetilde{R}_m), \tag{6.6}$$

$$\left.\frac{\partial \sigma(\widetilde{R}_p)}{\partial a}\right|_{a=0} = \frac{1}{2}(\sigma_m^2)^{-1/2}(-2\sigma_m^2 + 2\sigma_{im}) = \frac{\sigma_{im} - \sigma_m^2}{\sigma_m}. \tag{6.7}$$

The slope of the risk-return trade-off evaluated at point M, in market equilibrium, is

$$\left.\frac{\partial E(\widetilde{R}_p)/\partial a}{\partial \sigma(\widetilde{R}_p)/\partial a}\right|_{a=0} = \frac{E(\widetilde{R}_i) - E(\widetilde{R}_m)}{(\sigma_{im} - \sigma_m^2)/\sigma_m}. \tag{6.8}$$

The final insight is to realize that the slope of the opportunity set IMI' provided by the relationship between the risky asset and the market portfolio at point M must also be equal to the slope of the capital market line, $R_f M$.

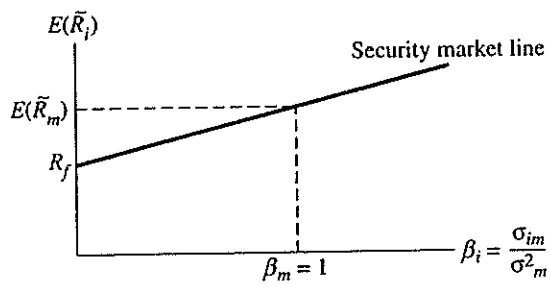


Figure 6.3 The capital asset pricing model.

As established in Chapter 5, the capital market line is also an equilibrium relationship. Given market efficiency, the tangency portfolio, M, must be the market portfolio where all assets are held according to their market value weights. Recall that the slope of the market line in Eq. (5.34) is

$$\frac{E(\tilde{R}_m) - R_f}{\sigma_m},$$

where σ_m is the standard deviation of the market portfolio. Equating this with the slope of the opportunity set at point M, we have

$$\frac{E(\tilde{R}_m) - R_f}{\sigma_m} = \frac{E(\tilde{R}_i) - E(\tilde{R}_m)}{(\sigma_{im} - \sigma_m^2)/\sigma_m}.$$

This relationship can be arranged to solve for $E(\tilde{R}_i)$ as follows:

$$E(\tilde{R}_i) = R_f + \left[E(\tilde{R}_m) - R_f\right]\frac{\sigma_{im}}{\sigma_m^2}. \tag{6.9}$$

Equation (6.9) is known as the *capital asset pricing model*, CAPM. It is shown graphically in Fig. 6.3, where it is also called the *security market line*. The required rate of return on any asset, $E(\tilde{R}_i)$ in Eq. (6.9), is equal to the risk-free rate of return plus a risk premium. The risk premium is the price of risk multiplied by the quantity of risk. In the terminology of the CAPM, the price of risk is the slope of the line, the difference between the expected rate of return on the market portfolio and the risk-free rate of return.[2] The quantity of risk is often called beta, β_i:

[2] Note that the CAPM terminology is somewhat different from that used in Chapter 5. Earlier, the equilibrium price of risk was seen to be the marginal rate of substitution between return and risk and was described as

$$\frac{E(\tilde{R}_m) - R_f}{\sigma_m}.$$

Using this definition for the price of risk, the quantity of risk is

$$\frac{\text{COV}(\tilde{R}_i, \tilde{R}_m)}{\sigma_m}.$$

Because σ_m, the standard deviation of the market, is assumed to be constant, it does not make much difference which terminology we adopt. Hereafter, risk will be β, and the market price of risk will be $\left[E(\tilde{R}_m) - R_f\right]$.

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} = \frac{\text{COV}(\widetilde{R}_i, \widetilde{R}_m)}{\text{VAR}(\widetilde{R}_m)}. \tag{6.10}$$

It is the covariance between returns on the risky asset, I, and market portfolio, M, divided by the variance of the market portfolio. The risk-free asset has a beta of zero because its covariance with the market portfolio is zero. The market portfolio has a beta of one because the covariance of the market portfolio with itself is identical to the variance of the market portfolio:

$$\beta_m = \frac{\text{COV}(\widetilde{R}_i, \widetilde{R}_m)}{\text{VAR}(\widetilde{R}_m)} = \frac{\text{VAR}(\widetilde{R}_m)}{\text{VAR}(\widetilde{R}_m)} = 1.$$

D. Properties of the CAPM

There are several properties of the CAPM that are important. First, in equilibrium, every asset must be priced so that its risk-adjusted required rate of return falls exactly on the straight line in Fig. 6.3, which is called the *security market line*. This means, for example, that assets such as I and J in Fig 5.20, which do not lie on the mean-variance efficient set, will lie exactly on the security market line in Fig. 6.3. This is true because not all the variance of an asset's return is of concern to risk-averse investors. As we saw in the previous chapter, investors can always diversify away all risk except the covariance of an asset with the market portfolio. In other words, they can diversify away all risk except the risk of the economy as a whole, which is inescapable (undiversifiable). Consequently, the only risk that investors will pay a premium to avoid is covariance risk. The total risk of any individual asset can be partitioned into two parts—systematic risk, which is a measure of how the asset covaries with the economy, and unsystematic risk, which is independent of the economy:

$$\text{total risk} = \text{systematic risk} + \text{unsystematic risk}. \tag{6.11}$$

Mathematical precision can be attached to this concept by noting that empirically the return on any asset is a linear function of market return plus a random error term $\widetilde{\varepsilon}_j$, which is independent of the market:

$$\widetilde{R}_j = a_j + b_j \widetilde{R}_m + \widetilde{\varepsilon}_j.$$

This equation contains three terms: a constant, a_j, which has no variance; a constant times a random variable, $b_j \widetilde{R}_m$; and a second random variable, $\widetilde{\varepsilon}_j$, which has zero covariance with $\widetilde{R}_m$. Using Properties 3 and 4 of random variables (given in Chapter 5), we can immediately write the variance of this relationship as

$$\sigma_j^2 = b_j^2 \sigma_m^2 + \sigma_\varepsilon^2. \tag{6.12}$$

Table 6.1 Risk and Return for Bayside Smokes and a 100-Stock Portfolio

	Annual Return (%)	Standard Deviation (%)	Beta
100-stock portfolio	10.9	4.45	1.11
Bayside Smokes	5.4	7.25	.71

The variance is total risk. It can be partitioned into systematic risk, $b_j^2\sigma_m^2$, and unsystematic risk, σ_ε^2. It turns out that b_j in the simple linear relationship between individual asset return and market return is exactly the same as β_j in the CAPM.[3]

If systematic risk is the only type of risk that investors will pay to avoid, and if the required rate of return for every asset in equilibrium must fall on the security market line, we should be able to go back to the example of Bayside Smokes Company and resolve the paradox introduced in Chapter 5. Table 6.1 summarizes the empirical findings. We know that if investors are risk averse, there should be a positive trade-off between risk and return. When we tried to use the standard deviation as a measure of risk for an individual asset, Bayside Smokes, in comparison with a well-diversified portfolio, we were forced to make the inappropriate observation that the asset with higher risk has a lower return. The difficulty was that we were using the wrong measure of risk. One cannot compare the variance of return on a single asset with the variance for a well-diversified portfolio. The variance of the portfolio will almost always be smaller. The appropriate measure of risk for a single asset is beta, its covariance with the market divided by the variance of the market. This risk is nondiversifiable and is linearly related to the rate of return—$E(R_i)$ in Eq. (6.9)—required in equilibrium. When we look at the appropriate measure of risk, we see that Bayside Smokes is *less risky* than the 100-stock portfolio, and we have the sensible result that lower risk is accompanied by lower return.

Table 6.2 shows the realized rates of return and the betas of many different assets between January 1945 and June 1970. The calculations are taken from an article by Modigliani and Pogue [1974] that used monthly observations. In most cases the risk-return relationships make sense. Consumer product companies such as Swift and Co., Bayside Smokes, and American Snuff are all less risky than the market portfolio (represented here by the NYSE index). On the other hand, steel, electronics, and automobiles are riskier. Fig. 6.4 plots the empirical relationship between risk (measured by beta) and return for the companies listed in Table 6.2. The linearity of the relationship appears to be reasonable, and the trade-off between risk and return is positive. A more thorough discussion of empirical tests of the CAPM will be given later in this chapter.

A second important property of the CAPM is that the measure of risk for individual assets is linearly additive when the assets are combined into portfolios. For example, if we put $a\%$ of our wealth into asset X, with systematic risk of β_x, and $b\%$ of our wealth into asset Y, with systematic risk of β_y, then the beta of the resulting portfolio, β_p, is simply the weighted average of the betas of the individual securities:

$$\beta_p = a\beta_x + b\beta_y. \tag{6.13}$$

[3] The interested reader is referred to Appendix C on linear regression at the end of the book for proof that the slope coefficient, b_j, equals

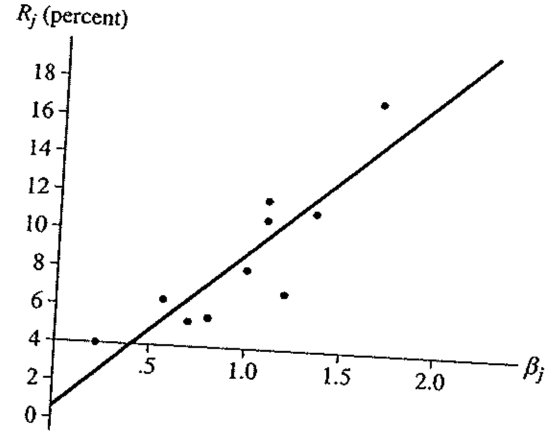
$$b_j = \mathrm{COV}(R_j, R_m)/\mathrm{VAR}(R_m) = \sigma_{jm}/\sigma_m^2$$

Table 6.2 Rates of Return and Betas for Selected Companies, 1945–1970

	Average Annual Return (%)	Standard Deviation (%)	Beta
City Investing Co.	17.4	11.09	1.67
Radio Corporation of America	11.4	8.30	1.35
Chrysler Corporation	7.0	7.73	1.21
Continental Steel Co.	11.9	7.50	1.12
100-stock portfolio	10.9	4.45	1.11
NYSE index	8.3	3.73	1.00
Swift and Co.	5.7	5.89	.81
Bayside Smokes	5.4	7.26	.71
American Snuff	6.5	4.77	.54
Homestake Mining Co.	4.0	6.55	.24

From F. Modigliani and G. Pogue, "An Introduction to Risk and Return," reprinted from *Financial Analysts Journal*, March–April 1974, 71.

Figure 6.4 An empirical security market line.



Proof of this follows from the definition of covariance and the properties of the mean and variance. The definition of the portfolio beta is

$$\beta_p = \frac{E\left\{[aX + bY - aE(X) - bE(Y)]\left[R_m - E(R_m)\right]\right\}}{\text{VAR}(R_m)}.$$

Rearranging terms, we have

$$\beta_p = \frac{E\left\{[a[X - E(X)] + b[Y - E(Y)]]\left[R_m - E(R_m)\right]\right\}}{\text{VAR}(R_m)}.$$

Next, we factor out a and b:

$$\beta_p = a\frac{E[(X - E(X))(R_m - E(R_m))]}{\text{VAR}(R_m)} + b\frac{E[(Y - E(Y))(R_m - E(R_m))]}{\text{VAR}(R_m)}.$$

Finally, using the definition of β,

$$\beta_p = a\beta_x + b\beta_y. \qquad \text{QED}$$

The fact that portfolio betas are linearly weighted combinations of individual asset betas is an extremely useful tool. All that is needed to measure the systematic risk of portfolios is the betas of the individual assets. It is not necessary to solve a quadratic programming problem (see Eqs. 5.26a and 5.26b) to find the efficient set.

It is worth reiterating the relationship between individual asset risk and portfolio risk. The correct definition of an individual asset's risk is its contribution to portfolio risk. Referring to Eq. (5.28), we see that the variance of returns for a portfolio of assets is

$$\text{VAR}(R_p) = \sigma^2(R_p) = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij},$$

which can be rewritten as[4]

$$\sigma^2(R_p) = \sum_{i=1}^{N}\left(\sum_{j=1}^{N} w_j \sigma_{ij}\right) = \sum_{i=1}^{N} w_i \text{COV}\left(R_i, R_p\right). \tag{6.14}$$

One could interpret

$$w_i \text{COV}(R_i, R_p) \tag{6.15}$$

as the risk of security i in portfolio p. However, at the margin, the change in the contribution of asset i to portfolio risk is simply

$$\text{COV}(R_i, R_p). \tag{6.16}$$

[4] To see that $\sum w_i(\sum w_j\sigma_{ij}) = \sum w_i\text{COV}(R_i, R_p)$, consider a simple three-asset example. Rewriting the left-hand side, we have

$$\begin{aligned}\sum w_i(\sum w_j\sigma_{ij}) &= w_1(w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13})\\ &+ w_2(w_1\sigma_{21} + w_2\sigma_{22} + w_3\sigma_{23})\\ &+ w_3(w_1\sigma_{31} + w_2\sigma_{32} + w_3\sigma_{33}).\end{aligned}$$

From the definition of covariance, we have

$$\begin{aligned}\text{COV}(R_1, R_p) &= [1 \quad 0 \quad 0]\begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13}\\ \sigma_{21} & \sigma_{22} & \sigma_{23}\\ \sigma_{31} & \sigma_{32} & \sigma_{33}\end{bmatrix}\begin{bmatrix}w_1\\ w_2\\ w_3\end{bmatrix}\\ &= w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13}.\end{aligned}$$

Then by multiplying by the weight in the first asset, we obtain

$$w_1\text{COV}(R_1, R_p) = w_1(w_1\sigma_{11} + w_2\sigma_{12} + w_3\sigma_{13}).$$

Finally, by repeating this procedure for each of the three assets, we can demonstrate the equality in Eq. (6.14).

Therefore covariance risk is the appropriate definition of risk since it measures the change in portfolio risk as we change the weighting of an individual asset in the portfolio.

Although the terms *systematic risk* and *undiversifiable risk* have arisen in the literature as synonyms for covariance risk, they are somewhat misleading. They rely on the existence of costless diversification opportunities and on the existence of a large market portfolio. The definition of covariance risk given above does not. It continues to be relevant, even when the market portfolio under consideration has few assets.

E. Use of the CAPM for Valuation: Single-Period Models with Uncertainty

Because it provides a quantifiable measure of risk for individual assets, the CAPM is an extremely useful tool for valuing risky assets. For the time being, let us assume that we are dealing with a single time period. This assumption was built into the derivation of the CAPM. We want to value an asset that has a risky payoff at the end of the period. Call this $\widetilde{P}_e$. It could represent the capital gain on a common stock or the capital gain plus a dividend. If the risky asset is a bond, it is the repayment of the principal plus the interest on the bond. The expected return on an investment in the risky asset is determined by the price we are willing to pay at the beginning of the time period for the right to the risky end-of-period payoff. If P_0 is the price we pay today, our risky return, $\widetilde{R}_j$, is

$$\widetilde{R}_j = \frac{\widetilde{P}_e - P_0}{P_0}. \tag{6.17}$$

The CAPM can be used to determine what the current value of the asset, P_0, should be. The CAPM is

$$E(R_j) = R_f + \left[E(R_m) - R_f\right] \frac{\text{COV}(R_j, R_m)}{\text{VAR}(R_m)},$$

which can be rewritten as

$$E(R_j) = R_f + \lambda \text{COV}(R_j, R_m), \quad \text{where} \quad \lambda = \frac{E(R_m) - R_f}{\text{VAR}(R_m)}. \tag{6.18}$$

Note that λ can be described as the market price per unit risk. From Eq. (6.17) and the properties of the mean, we can equate the expected return from Eq. (6.17) with the expected return in Eq. (6.18):

$$\frac{E(P_e) - P_0}{P_0} = R_f + \lambda \text{COV}(R_j, R_m).$$

We can now interpret P_0 as the equilibrium price of the risky asset. Rearranging the above expression, we get

$$P_0 = \frac{E(P_e)}{1 + R_f + \lambda \text{COV}(R_j, R_m)}, \tag{6.19}$$

which is often referred to as the *risk-adjusted rate of return valuation formula*. The numerator is the expected end-of-period price for the risky asset, and the denominator can be thought of as a discount rate. If the asset has no risk, then its covariance with the market will be zero, and the appropriate one-period discount rate is $(1 + R_f)$. For assets with positive systematic risk, a risk premium, $\lambda\text{COV}(R_j, R_m)$, is added to the risk-free rate so that the discount rate is risk adjusted.

An equivalent approach to valuation is to deduct a risk premium from $E(P_e)$ in the numerator, then discount at $(1 + R_f)$. The covariance between the risky asset and the market can be rewritten as

$$\begin{aligned}\text{COV}(R_j, R_m) &= \text{COV}\left[\frac{P_e - P_0}{P_0}, R_m\right] \\ &= E\left[\left(\frac{P_e - P_0}{P_0} - \frac{E(P_e) - P_0}{P_0}\right)(R_m - E(R_m))\right] \\ &= \frac{1}{P_0}\text{COV}(P_e, R_m).\end{aligned}$$

By substituting this into the risk-adjusted rate of return equation, Eq. (6.19),

$$P_0 = \frac{E(P_e)}{1 + R_f + \lambda(1/P_0)\text{COV}(P_e, R_m)},$$

we can derive the *certainty equivalent valuation formula* from

$$P_0 = \frac{E(P_e) - \lambda\text{COV}(P_e, R_m)}{1 + R_f}. \tag{6.20}$$

The risk-adjusted rate of return and the certainty equivalent approaches are equivalent for one-period valuation models. It is important to realize that in both cases value does not depend on the utility preferences of individuals. All one needs to know in order to determine value is the expected end-of-period payoff, the quantity of risk provided by the asset, the risk-free rate, and the price of risk (which are market-determined variables). Consequently, individuals who perceive the same distribution of payoffs for a risky asset will price it exactly the same way regardless of their individual utility functions. The separation of valuation from attitudes toward risk is a consequence of two-fund separation. This was discussed in Chapter 5, Section E.5.

F. Applications of the CAPM for Corporate Policy

In the second half of this book, when we focus on corporate policy, these one-period valuation models will be used to develop decision-making rules for the selection of investment projects by the firm, for measurement of the firm's cost of capital, and for capital structure (optimal debt/equity ratio) decisions. However, for the sake of curiosity, we shall take a quick look at the implications of the CAPM for some corporate policy decisions, assuming that our firm has no debt and that there are no corporate or personal taxes. The complex results in a world with debt and taxes are left to Chapter 15.

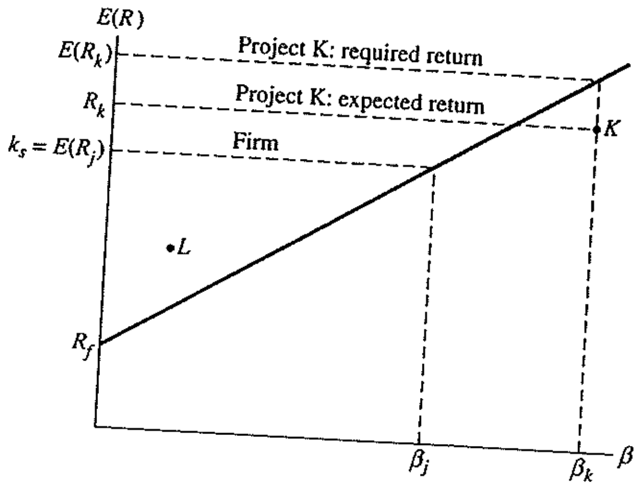


Figure 6.5 The cost of equity using the CAPM.

The cost of equity capital for a firm is given directly by the CAPM. After all, the company's beta is measured by calculating the covariance between the return on its common stock and the market index. Consequently, the beta measures the systematic risk of the common stock, and if we know the systematic risk, we can use the CAPM to determine the required rate of return on equity. Equation (6.21) is the capital asset pricing model:

$$E(R_j) = R_f + \left[E(R_m) - R_f\right]\beta_j. \tag{6.21}$$

If it is possible to estimate the systematic risk of a company's equity as well as the market rate of return, the $E(R_j)$ is the required rate of return on equity, that is, the cost of equity for the firm. If we designate the cost of equity as k_s, then

$$E(R_j) = k_s.$$

This is shown in Fig. 6.5. As long as projects have the same risk as the firm, then k_s may also be interpreted as the minimum required rate of return on new capital projects.

But what if the project has a different risk from the firm as a whole? Then all that is necessary is to estimate the systematic risk of the project and use the appropriate rate of return, $E(R_k)$. For example, in Fig. 6.5 the expected rate of return on project K, R_k, is higher than the cost of equity for the firm, $E(R_j)$. But the project also is riskier than the firm because it has greater systematic risk. If the managers of the firm were to demand that it earn the same rate as the firm $[k_s = E(R_j)]$, the project would be accepted since its expected rate of return, R_k, is greater than the firm's cost of equity. However, this would be incorrect. The market requires a rate of return, $E(R_k)$, for a project with systematic risk of β_k, but the project will earn less. Therefore since $R_k < E(R_k)$, the project is clearly unacceptable. (Is project L acceptable? Why?)

Because the CAPM allows decision makers to estimate the required rate of return for projects of different risk, it is an extremely useful concept. Although we have assumed no debt or taxes in the above simple introduction, Chapter 15 will show how the model can be extended to properly conceptualize more realistic capital budgeting and cost of capital decisions.

G. Extensions of the CAPM

Virtually every one of the assumptions under which the CAPM is derived is violated in the real world. If so, then how good is the model? There are two parts to this question: (1) Is it possible to extend the model to relax the unrealistic assumptions without drastically changing it? (2) How well does the model stand up to empirical testing? The first part is the subject of this section of the chapter. Surprisingly, the model is fairly resilient to various extensions of it.

1. No Riskless Asset

First, how will the model change if investors cannot borrow and lend at the risk-free rate? In other words, how is the CAPM affected if there is no risk-free asset that has constant returns in every state of nature? This problem was solved by Black [1972]. His argument is illustrated in Fig. 6.6. Portfolio M is identified by the investors as the market portfolio that lies on the efficient set.[5] Now, suppose that we can identify all portfolios that are uncorrelated with the true market portfolio.[6] This means that their returns have zero covariance with the market portfolio, and they have the same systematic risk (i.e., they have zero beta). Because they have the same systematic risk, each must have the same expected return. Portfolios A and B in Fig. 6.6 are both uncorrelated with the market portfolio M and have the same expected return $E(R_z)$. However, only one of them, portfolio B, lies on the opportunity set. It is the *minimum-variance zero-beta portfolio,* and it is unique. Portfolio A also has zero beta, but it has a higher variance and therefore does not lie on the minimum-variance opportunity set.

We can derive the slope of the line $E(R_z)M$ by forming a portfolio with $a\%$ in the market portfolio and $(1-a)\%$ in the minimum-variance zero-beta portfolio. The mean and standard deviation of such a portfolio can be written as follows:

$$E(R_p) = aE(R_m) + (1-a)E(R_z),$$

$$\sigma(R_p) = \left[a^2\sigma_m^2 + (1-a)^2\sigma_z^2 + 2a(1-a)r_{zm}\sigma_z\sigma_m\right]^{1/2}.$$

[5] Note, however, that the extension of the CAPM that follows can be applied to *any* efficient portfolio, not just the market portfolio.

[6] As an example of how to calculate the vector of weights in a world with only two assets, see Problem 6.13. For portfolios with many assets, we are interested in identifying the portfolio that (a) has zero covariance with the market portfolio and (b) has the mimimum variance. The solution will be the vector of the weights that satisfies the following quadratic programming problem:

$$\text{MIN}\,\sigma_p^2 = W_1' \sum W_1$$

$$\text{Subject to } W_1' \sum W_m = \sigma_{1m} = 0$$

$$W_1' e = 1,$$

where σ_p^2 = the variance of the zero-beta portfolio,
W_1' = the row vector of weights in the mimimum-variance zero-beta portfolio (W_1 is a column vector with the same weights),
$\sum$ = the variance/covariance matrix for all N assets in the market,
W_m = the vector of weights in the market portfolio,
σ_{1m} = the covariance between the zero-beta portfolio and the market—which must always equal zero,
e = a column vector of ones.

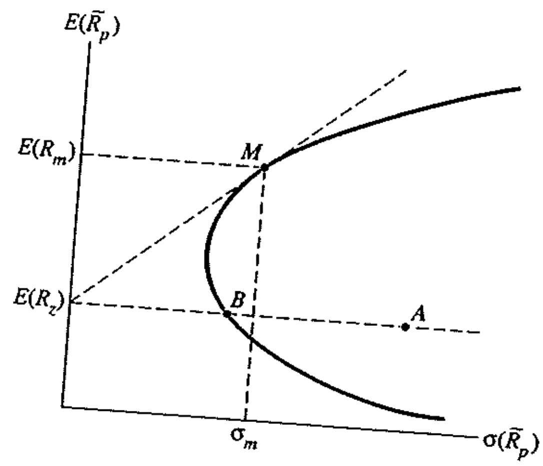


Figure 6.6 The capital market line with no risk-free rate.

But since the correlation, r_{zm}, between the zero-beta portfolio and the market portfolio is zero, the last term drops out. The slope of a line tangent to the efficient set at point M, where 100% of the investor's wealth is invested in the market portfolio, can be found by taking the partial derivatives of the above equations and evaluating them where $a = 1$. The partial derivative of the mean portfolio return is

$$\frac{\partial E(R_p)}{\partial a} = E(R_m) - E(R_z),$$

and the partial derivative of the standard deviation is

$$\frac{\partial \sigma(R_p)}{\partial a} = \frac{1}{2}\left[a^2\sigma_m^2 + (1-a)^2\sigma_z^2\right]^{-1/2}\left[2a\sigma_m^2 - 2\sigma_z^2 + 2a\sigma_z^2\right].$$

Taking the ratio of these partials and evaluating where $a = 1$, we obtain the slope of the line $E(R_z)M$ in Fig. 6.6:

$$\frac{\partial E(R_p)/\partial a}{\partial \sigma(R_p)/\partial a} = \frac{E(R_m) - E(R_z)}{\sigma_m}. \tag{6.22}$$

Furthermore, since the line must pass through the point $\left[E(R_m), \sigma(R_m)\right]$, the intercept of the tangent line must be $E(R_z)$. Consequently, the equation of the line must be

$$E(R_p) = E(R_z) + \left[\frac{E(R_m) - E(R_z)}{\sigma_m}\right]\sigma_p. \tag{6.23}$$

This is exactly the same as the capital market line in Eq. (5.34) except that the expected rate of return on the zero-beta portfolio, $E(R_z)$, has replaced the risk-free rate.

Given the above result, it is not hard to prove that the expected rate of return on *any* risky asset, whether or not it lies on the efficient set, must be a linear combination of the rate of return on the zero-beta portfolio and the market portfolio. To show this, recall that in equilibrium the slope of a line tangent to a portfolio composed of the market portfolio and any other asset at the point represented by the market portfolio must be equal to Eq. (6.8):

$$\left.\frac{\partial E(R_p)/\partial a}{\partial \sigma(R_p)/\partial a}\right|_{a=0} = \frac{E(R_i) - E(R_m)}{(\sigma_{im} - \sigma_m^2)/\sigma_m}.$$

If we equate the two definitions of the slope of a line tangent to point M, that is, if we equate (6.8) and (6.22), we have

$$\frac{E(R_m) - E(R_z)}{\sigma_m} = \frac{[E(R_i) - E(R_m)]\,\sigma_m}{\sigma_{im} - \sigma_m^2}.$$

Solving for the required rate of return on asset i, we have

$$E(R_i) = (1 - \beta_i)E(R_z) + \beta_i E(R_m), \tag{6.24}$$

where

$$\beta_i = \sigma_{im}/\sigma_m^2 = \text{COV}(R_i, R_m)/\sigma_m^2. \tag{6.25}$$

Equation (6.24) shows that the expected rate of return on any asset can be written as a linear combination of the expected rate of return of any two assets—the market portfolio and the unique minimum-variance zero-beta portfolio (which is chosen to be uncorrelated with the market portfolio). Interestingly, the weight to be invested in the market portfolio is the beta of the ith asset. If we rearrange (6.24), we see that it is exactly equal to the CAPM (Eqs. 6.9 and 6.21) except that the expected rate of return on the zero-beta portfolio has replaced the rate of return on the risk-free asset:

$$E(R_i) = E(R_z) + [E(R_m) - E(R_z)]\,\beta_i. \tag{6.26}$$

The upshot of this proof is that the major results of the CAPM do not require the existence of a pure riskless asset. Beta is still the appropriate measure of systematic risk for an asset, and the linearity of the model still obtains. The version of the model given by Eq. (6.26) is usually called the *two-factor model*.[7]

2. The Existence of Nonmarketable Assets

Suppose that the cost of transacting in an asset is infinite or that by law or regulation the asset is not marketable. Perhaps the most important example of such an asset is human capital. You can rent your skills in return for wages, but you cannot sell yourself or buy anyone else. Slavery is forbidden. This has the effect of introducing a nondiversifiable asset into your portfolio—your human asset capital. Because you cannot divide up your skills and sell them to different investors, you are forced into making portfolio decisions where you are constrained to hold a large risky

[7] One limitation of the two-factor model is that it relies rather heavily on the assumption that there are no short sales constraints. In general, zero-beta portfolios would have to be composed of both long and short positions of risky assets. Ross [1977] has shown that in a world with short sales restrictions and no riskless asset the linear CAPM is invalid. Thus to obtain the CAPM in a linear form (Eqs. 6.9 and 6.25) we require either (1) a risk-free asset that can be freely short sold or (2) no constraints on short sales.

component of your wealth in the form of your own human capital. What impact does this have on portfolio decisions and the CAPM?

We saw earlier that if there are no transaction costs and if all assets are perfectly divisible, two-fund separation obtains (see Chapter 5). Every investor, regardless of the shape of his or her indifference curve, will hold one of two assets: the risk-free asset or the market portfolio. Of course, casual empiricism tells us that this is not what actually happens. People do hold different portfolios of risky assets. There are many reasons why this may be true, and the existence of nonmarketable assets is a good possibility.

Mayers [1972] shows that when investors are constrained to hold nonmarketable assets that have risky (dollar) rates of return, R_H, the CAPM takes the following form:

$$E(R_j) = R_f + \lambda\left[V_m \text{COV}(R_j, R_m) + \text{COV}(R_j, R_H)\right], \tag{6.27}$$

where

$$\lambda = \frac{E(R_m) - R_f}{V_m \sigma_m^2 + \text{COV}(R_m, R_H)},$$

V_m = the current market value of all marketable assets.

R_H = the total dollar return on all nonmarketable assets.

In this version of the model, λ may be interpreted as the market price per unit risk where risk contains not only the market variance, σ_m^2, but also the covariance between the rate of return on marketable assets and the aggregate dollar return on nonmarketable assets. This result is obtained by first deriving an individual's demand curves for holding marketable assets, then aggregating them to obtain Eq. (6.26), which is the return on a marketable asset required by the market equilibrium. There are three important implications. First, individuals will hold different portfolios of risky assets because their human capital has differing amounts of risk. Second, the market equilibrium price of a risky asset may still be determined independently of the shape of the individual's indifference curves. This implies that the separation principle still holds. There is still an objectively determined market price of risk that is independent of individual attitudes toward risk. No variable in Eq. (6.26) is subscripted for the preference of the ith individual. Both the price of risk and the amount of risk depend only on properties of the jth asset, the portfolio of all marketable assets, and the portfolio of aggregated nonmarketable assets. Third, the appropriate measure of risk is still the covariance, but we must now consider the covariance between the jth risky asset and two portfolios, one composed of marketable and a second of nonmarketable assets.[8]

3. The Model in Continuous Time

Merton [1973] has derived a version of the CAPM that assumes (among other things) that trading takes place continuously over time, and that asset returns are distributed lognormally. If the risk-free rate of interest is nonstochastic over time, then (regardless of individual preferences, the distribution of individuals' wealth, or their time horizon) the equilibrium returns must satisfy

$$E(R_i) = r_f + \left[E(R_m) - r_f\right]\beta_i. \tag{6.28}$$

[8] See Fama and Schwert [1977] for an empirical test of the model set forth by Mayers.

Eq. (6.28) is the continuous-time analogy to the CAPM. In fact, it is exactly the same as the CAPM except that instantaneous rates of return have replaced rates of return over discrete intervals of time, and the distribution of returns is lognormal instead of normal.

If the risk-free rate is stochastic, investors are exposed to another kind of risk, namely, the risk of unfavorable shifts in the investment opportunity set. Merton shows that investors will hold portfolios chosen from three funds: the riskless asset, the market portfolio, and a portfolio chosen so that its returns are perfectly negatively correlated with the riskless asset. This model exhibits three-fund separation. The third fund is necessary to hedge against unforeseen changes in the future risk-free rate. The required rate of return on the jth asset is

$$E(R_j) = r_f + \gamma_1 \left[E(R_m) - r_f\right] + \gamma_2 \left[E(R_N) - r_f\right], \tag{6.29}$$

where

$R_N =$ the instantaneous rate of return on a portfolio that has perfect negative correlation with the riskless asset,

$$\gamma_1 = \frac{\beta_{jm} - \beta_{jN}\beta_{Nm}}{1 - \rho^2_{Nm}}, \qquad \gamma_2 = \frac{\beta_{jn} - \beta_{jm}\beta_{Nm}}{1 - \rho^2_{Nm}},$$

$\rho_{Nm} =$ the correlation between portfolio N and the market portfolio, M,

$$\beta_{ik} = \frac{\text{COV}(R_i, R_k)}{\sigma^2_k}.$$

Merton argues that the sign of γ_2 will be negative for high-beta assets and positive for low-beta assets. As we shall see in the next section, which discusses the empirical tests of the CAPM, Merton's argument is consistent with the empirical evidence.

4. The Existence of Heterogeneous Expectations and Taxes

If investors do not have the same information about the distribution of future returns, they will perceive different opportunity sets and will obviously choose different portfolios. Lintner [1969] has shown that the existence of heterogeneous expectations does not critically alter the CAPM except that expected returns and covariances are expressed as complex weighted averages of investor expectations. However, if investors have heterogeneous expectations, the market portfolio is not necessarily efficient. This makes the CAPM nontestable. In fact, as we shall see when we discuss Roll's critique later in this chapter, the only legitimate test of the CAPM is a joint test to determine whether or not the market portfolio is efficient.

No one has investigated the equilibrium model in a world with personal as well as corporate taxes. However, Brennan [1970] has investigated the effect of differential tax rates on capital gains and dividends. Although he concludes that beta is the appropriate measure of risk, his model includes an extra term that causes the expected return on an asset to depend on dividend yield as well as systematic risk:

$$E(R_j) = \gamma_1 R_f + \gamma_2 \beta_j + \gamma_3 DY_j \tag{6.30}$$

where

$DY_j =$ the dividend yield on asset j.

We shall leave a complete discussion of the Brennan model to Chapter 16, which will cover the theory and empirical evidence related to the corporate dividend policy decision. For now it is sufficient to note that Brennan's model predicts that higher rates of return will be required on assets with higher dividend yields. In other words, investors do not like dividends because they must pay ordinary income tax rates on dividends but only capital gains rates on stock price increases.

H. Empirical Tests of the CAPM

We begin the discussion of empirical tests with Table 6.3, which clearly shows that riskier (well-diversified) portfolios have higher returns over very long periods of time. *Geometric returns* are measured as the geometric average holding period return, r_g, calculated as follows for a portfolio that is reweighted at the beginning of each period:

$$r_g = [\prod(1 + r_{pt})]^{1/N} - 1 \tag{6.31}$$

Arithmetic returns for the same portfolio are

$$r_a = 1/N[\sum(1 + r_{pt})] - 1 \tag{6.32}$$

The CAPM is a simple linear model that is expressed in terms of expected returns and expected risk. In its ex ante form, we have

$$E(R_j) = R_f + [E(R_m) - R_f]\beta_j. \tag{6.33}$$

If the CAPM is a better model of the reward-risk trade-off for individual securities, it should be an improvement over Table 6.3 because it predicts that beta, not the standard deviation, should be a better measure of risk. Although many of the aforementioned extensions of the CAPM model support its simple linear form, others suggest that it may not be linear, that factors other than beta are needed to explain $E(R_j)$, or that R_f is not the appropriate risk-free rate. Therefore with so many alternative possibilities a great deal of energy has been devoted to the empirical question: How well does the model fit the data? In fact, researchers have been working on tests of the CAPM for nearly 40 years, and no conclusive evidence has been published to date—the jury is still out.

Table 6.3 Annual Average Market Data U.S. 1963–2002 (Percent)

	Geometric Return	Arithmetic Return	Geometric Standard Deviation	Arithmetic Standard Deviation
Stocks of smaller companies	19.6	25.2	42.6	41.8
S & P 500	10.5	11.9	16.9	16.6
Long-term corporate bonds	8.4	8.5	3.2	3.2
Long-term U.S. government bonds	7.0	7.1	3.1	3.1
Short-term U.S. Treasury bills	6.3	6.3	2.5	2.5

Source: Monitor Group analysis.

There have been numerous empirical tests of the CAPM, so many in fact that it would be fruitless to mention all of them. Also, the literature is interwoven with many serious and difficult econometric problems that must be confronted in order to provide the best empirical tests of the model.[9] However, in the opinion of the authors, the tests of the CAPM summarized below represent the best of the work that has been done to date.

The first step necessary to empirically test the theoretical CAPM is to transform it from expectations or ex ante form (expectations cannot be measured) into a form that uses observed data. This can be done by assuming that the rate of return on any asset is a *fair game*.[10] In other words, on average the realized rate of return on an asset is equal to the expected rate of return. We can write the fair game as follows:

$$R_{jt} = E(R_{jt}) + \beta_j \delta_{mt} + \varepsilon_{jt}, \tag{6.34}$$

where

$$\begin{aligned} \delta_{mt} &= R_{mt} - E(R_{mt}), \\ E(\delta_{mt}) &= 0, \\ \varepsilon_{jt} &= \text{ a random-error term,} \\ E(\varepsilon_{jt}) &= 0 \\ \text{COV}(\varepsilon_{jt}, \delta_{mt}) &= 0, \\ \beta_j &= \text{COV}(R_{jt}, R_{mt})/\text{VAR}(R_{mt}) \end{aligned}$$

Equation (6.34) is seen to be a fair game because if we take the expectation of both sides, the average realized return is equal to the expected return. In other words, on average you get the return you expected:

$$E(R_{jt}) = E(R_{jt}).$$

By substituting $E(R_j)$ from the CAPM into Eq. (6.34), we obtain

$$\begin{aligned} R_{jt} &= R_{ft} + \left[E(R_{mt}) - R_{ft}\right]\beta_j + \beta_j\left[R_{mt} - E(R_{mt})\right] + \varepsilon_{jt} \\ &= R_{ft} + (R_{mt} - R_{ft})\beta_j + \varepsilon_{jt}. \end{aligned}$$

Finally, by subtracting R_{ft} from both sides, we have

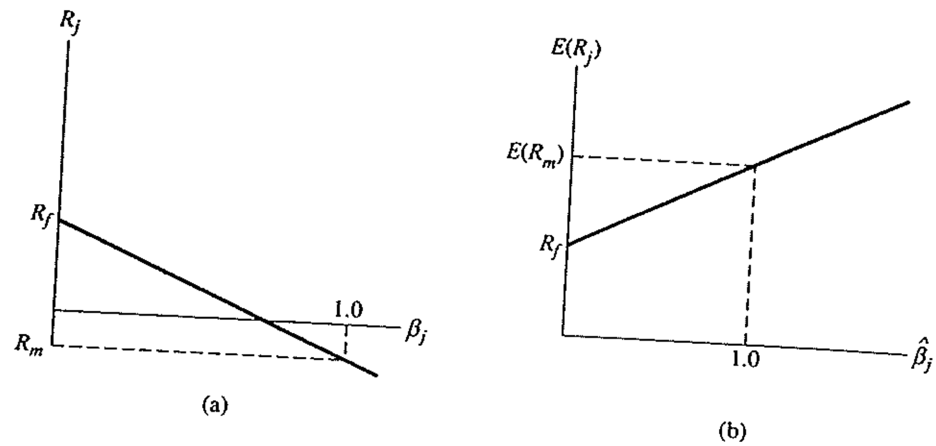
$$R_{jt} - R_{ft} = (R_{mt} - R_{ft})\beta_j + \varepsilon_{jt}, \tag{6.35}$$

which is the ex post form of the CAPM. We derived it by simply assuming that returns are normally distributed and that capital markets are efficient in a fair game sense. Now we have an empirical version of the CAPM that is expressed in terms of ex post observations of return data instead of ex ante expectations.

[9] For papers that discuss some of the econometric problems involved in testing the CAPM, the reader is referred to Miller and Scholes [1972], Roll [1977, 1981], Scholes and Williams [1977], Dimson [1979], and Gibbons [1982].

[10] Chapter 10 explains the theory of efficient capital markets that describes a fair game at length. Also, empirical evidence is presented that suggests that the market is in fact a fair game.

Figure 6.7 (a) Ex post CAPM; (b) ex ante CAPM.



One important difference between the ex post empirical model and the ex ante theoretical model is that the former can have a negative slope, whereas the latter cannot. After the fact we may have experienced a state of nature where the market rate of return was negative. When this happens the empirical security market line will slope downward as in Fig. 6.7(a). On the other hand, the theoretical CAPM always requires the ex ante expected return on the market to be higher than the risk-free rate of return as shown in Fig. 6.7(b). This is because prices must be established in such a way that riskier assets have higher expected rates of return. Of course, it may turn out that after the fact their return was low or negative, but that is what is meant by risk. If a risky asset has a beta of 2.0, it will lose roughly 20% when the market goes down by 10%.

When the CAPM is empirically tested, it is usually written in the following form:

$$R'_{pt} = \gamma_0 + \gamma_1 \beta_p + \varepsilon_{pt}, \tag{6.36}$$

where

$$\gamma_1 = R_{mt} - R_{ft},$$

$$R'_{pt} = \text{the excess return on portfolio } p = (R_{pt} - R_{ft}).$$

This is the same as Eq. (6.35) except that a constant term, γ_0, has been added.

Exactly what predictions made by the CAPM are tested in Eq. (6.36)?

1. The intercept term, γ_0, should not be significantly different from zero. If it is different from zero, then there may be something "left out" of the CAPM that is captured in the empirically estimated intercept term.
2. Beta should be the only factor that explains the rate of return on the risky asset. If other terms such as residual variance, dividend yield, price/earnings ratios, firm size, or beta squared are included in an attempt to explain return, they should have no explanatory power.
3. The relationship should be linear in beta.
4. The coefficient of beta, γ_1, should be equal to $(R_{mt} - R_{ft})$.

5. When the equation is estimated over very long periods of time, the rate of return on the market portfolio should be greater than the risk-free rate. Because the market portfolio is riskier, on average it should have a higher rate of return.

The major empirical tests of the CAPM were published by Friend and Blume [1970], Black, Jensen, and Scholes [1972], Miller and Scholes [1972], Blume and Friend [1973], Blume and Husick [1973], Fama and Macbeth [1973], Basu [1977], Reinganum [1981b], Litzenberger and Ramaswamy [1979], Banz [1981], Gibbons [1982], Stambaugh [1982], Shanken [1985b], Fama and French [1992], and Kothari, Shanken, and Sloan [1995]. Most of the studies use monthly total returns (dividends are reinvested) on listed common stocks as their database. A frequently used technique is to estimate the betas of every security during a five-year holding period, by computing the covariance between return on the security and a market index that is usually an equally weighted index of all listed common stocks. The securities are then ranked by beta and placed into N portfolios (where N is usually 10, 12, or 20). By grouping the individual securities into large portfolios chosen to provide the maximum dispersion in systematic risk, it is possible to avoid a good part of the measurement error in estimating betas of individual stocks. Next, the portfolio betas and returns are calculated over a second five-year period and a regression similar to Eq. (6.33) is run.

With few exceptions, the empirical studies prior to Fama and French [1992] agree on the following conclusions:

1. The intercept term, γ_0, is significantly different from zero, and the slope, γ_1, is less than the difference between the return on the market portfolio minus the risk-free rate.[11] The implication is that low-beta securities earn more than the CAPM would predict and high-beta securities earn less.
2. Versions of the model that include a squared-beta term or unsystematic risk find that at best these explanatory factors are used only in a small number of the time periods sampled. Beta dominates them as a measure of risk.
3. The simple linear empirical model of Eq. (6.36) fits the data best. It is linear in beta. Also, over long periods of time the rate of return on the market portfolio is greater than the risk-free rate (i.e., $\gamma_1 > 0$).
4. Factors other than beta are successful in explaining that portion of security returns not captured by beta. Basu [1977] found that low price/earnings portfolios have rates of return higher than could be explained by the CAPM. Banz [1981] and Reinganum [1981b] found that the size of a firm is important. Smaller firms tend to have high abnormal rates of return. Litzenberger and Ramaswamy [1979] found that the market requires higher rates of return on equities with high dividend yields. Keim [1983, 1985] reports seasonality in stock returns—a January effect, where most of excess returns are earned in one month. Fama and French [1992] conclude that market capitalization (a measure of size) and the ratio of the book to the market value of equity should replace beta altogether.

Figure 6.8 shows the average monthly returns on 10 portfolios versus their systematic risk for the 35-year period 1931–1965 (taken from the Black-Jensen-Scholes study [1972]). The results shown

[11] Empirical studies have used 90-day Treasury bills as a proxy for the risk-free rate, and they have also laboriously calculated the return on the zero-beta portfolio. Either approach results in an intercept term significantly different from zero.

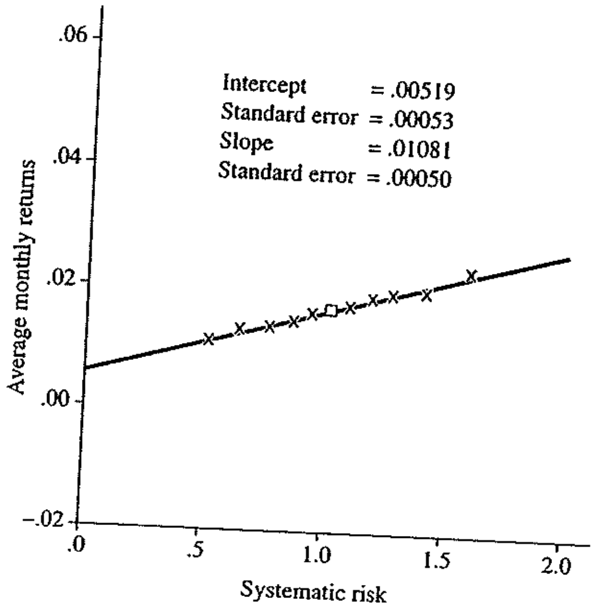


Figure 6.8 Average monthly returns vs. systematic risk for 10 portfolios, 1931–1965. (From *Studies in the Theory of Capital Markets*, edited by Michael C. Jensen. Copyright ©1972 by Praeger Publishers, Inc. Reprinted with permission of Holt, Rinehart, and Winston.)

here are typical. The empirical market line is linear with a positive trade-off between return and risk, but the intercept term is significantly different from zero. In fact, it is 9.79 standard deviations away. This forces us to reject the CAPM, given the empirical techniques of the previously mentioned studies. In addition, the ability of the other variables such as price/earnings ratios to explain the portion of returns that are unexplained by the CAPM suggests either (1) that the CAPM is misspecified and requires the addition of factors other than beta to explain security returns or (2) that the problems in measuring beta are systematically related to variables such as firm size. Work that is consistent with this second point of view has been published by Rosenberg and Marathe [1977], who find that beta can be predicted much better if variables such as dividend yield, trading volume, and firm size are added to the predictive model. Roll [1981] suggests that infrequent trading of shares in small firms may explain much of the measurement error in estimating betas.

Gibbons [1982], Stambaugh [1982], and Shanken [1985b] test the CAPM by first assuming that the market model is true—that is, the return on the ith asset is a linear function of a market portfolio proxy such as an equally weighted market portfolio:

$$R_{it} = \alpha_i + \beta_i R_{mt} + \varepsilon_{it}. \tag{6.37}$$

The market model, Eq. (6.37), is merely a statistical statement. It is not the CAPM. The CAPM—for example, Black's [1972] two-factor version—actually requires the intercept term, $E(R_z)$, in Eq. (6.37) to be the same for all assets. The two-factor CAPM is true across all assets at a point in time,

$$E(R_i) = E(R_z) + \left[E(R_m) - E(R_z)\right]\beta_i. \tag{6.38}$$

Table 6.4 Portfolio Monthly Rate of Return, Sorted on Size and Beta 1963–1990

	1A	1B	2	3	4	5	6	7	8	9	10A	10B
Portfolios Formed on Size												
Average monthly return	1.64	1.16	1.29	1.24	1.25	1.29	1.17	1.07	1.10	0.95	0.88	0.90
Beta	1.44	1.44	1.39	1.34	1.33	1.24	1.22	1.16	1.08	1.02	0.95	0.90
ln(market equity)	1.98	3.18	3.63	4.10	4.50	4.89	5.30	5.73	6.24	6.82	7.39	8.44
Number of companies	722	189	236	170	144	140	128	125	119	114	60	64
Portfolios Formed on Beta												
Average monthly return	1.20	1.20	1.32	1.26	1.31	1.30	1.30	1.23	1.23	1.33	1.34	1.18
Beta	0.81	0.79	0.92	1.04	1.13	1.19	1.26	1.32	1.14	1.52	1.63	1.73
ln (market equity)	1.98	3.18	3.63	4.10	4.50	4.89	5.30	5.73	6.24	6.82	7.39	8.44
Number of companies	116	80	185	181	179	182	185	205	227	267	165	291

Source: Fama and French [1992].

Gibbons [1982] points out that the two-factor CAPM implies the following constraint on the intercept of the market model:

$$\alpha_i = E(R_z)(1 - \beta_i) \tag{6.39}$$

for all securities during the same time interval. When he tests restriction (6.39), he finds that it is violated and that the CAPM must be rejected.

Fama and French [1992] published a landmark study on the cross-sectional relationship between return and risk, as a test of the CAPM. They used larger databases then ever before, starting with daily individual stock returns from 1963 to 1990 for all NYSE and AMEX listed stocks, then adding all NASDAQ-listed stocks from 1973 to 1990. Cognizant of the strong negative correlation between size and beta (−0.988), they designed an empirical test that carefully separated the two. Table 6.4 shows simple sorts based on size (measured as the natural logarithm of the equity market capitalization of a company) and beta (estimated using the value-weighted portfolio of NYSE, AMEX, and after 1973, NASDAQ stocks). The results clearly show that when portfolios are arranged by size there is a visible relationship between monthly returns and both size and beta, but when the portfolios are arranged by beta, the relationship becomes tenuous.

Therefore, Fama and French ran a two-pass sort to attempt to separate the effect of beta from size. The results are shown in Table 6.5. The numbers in each cell are the percentage average monthly returns. If we scan down the columns, we see the relationship between return and company size. It is fairly clear. If we read across the rows, we are seeing the relationship between return and beta, holding size constant. For the small and the large equity rows, the relationship between beta and return goes in the wrong direction and is not very strong in the remaining rows. Finally, Fama and French ran multiple regressions with the individual stock returns as the dependent variable. Beta was not statistically significantly related to the average returns of individual securities, either by itself, or when combined with size in a multiple regression. The strongest model did not include beta at all. It explained the return as a negative function of size (measured as the natural logarithm of market capitalization) and a positive function of the logarithm of the ratio of the book to the market value of equity. However, when they exclude the NASDAQ stocks and extend their data back to include the period 1941–1990, then beta is significant and positively related to returns both for portfolios and for individual stocks.

Table 6.5 Two-Pass Sort by Size and Beta Average Monthly Returns, 1963–1990 (percent)

	Low Beta	Third Decile	Average of Fifth and Sixth	Eighth Decile	High Beta
Small equity	1.71	1.79	1.50	1.63	1.42
Third decile	1.12	1.17	1.19	1.36	0.76
Average of fifth and sixth deciles	1.21	1.33	1.25	1.16	1.05
Eighth decile	1.09	1.37	1.13	1.02	0.94
Large equity	1.01	1.10	0.91	0.71	0.56

Source: Fama and French [1992]. (© 1992 Blackwell Publishing. Reprinted with permission.)

The Fama and French [1992] results were soon followed by others. Roll and Ross [1994] showed that even small departures of the index portfolio from ex post market efficiency can easily result in empirical results that show no relationship between beta and average cross-sectional returns. Kothari, Shanken, and Sloan [1995] study the same relationship between beta, size, and the ratio of book to market value of equity. They conclude that "examination of the cross-section of expected returns reveals economically and statistically significant compensation (about 6–9% per annum) for beta risk." They note that annual returns are used to avoid the seasonality of returns, there is a significant linear relationship between returns and beta between 1941 and 1990, that size is also related to returns but the incremental economic contribution is small (less than 1 percent), and that the book to market results are the result of survivorship bias in the Compustat database and are not economically significant.[12]

Fama and French [1996] update their work and present a three-factor model that, when fit to data for NYSE, AMEX, and NASDAQ stock returns for 366 months from July 1963 to December 1993, provides the best explanation of excess returns. Their model, written below, says that the excess return of the ith security over the risk-free rate is a linear function of three factors: the excess return of the market portfolio over the risk-free rate, the difference between the returns on a portfolio of small stocks and large stocks, $E(SMB)$, and the difference between the returns on a portfolio of high and low book-to-market stocks, $E(HML)$:

$$E(R_i) - R_f = b_i[E(R_m) - R_f] + s_i E(SMB) + h_i E(HML). \tag{6.40}$$

The debate concerning the empirical validity of the CAPM continues today and may be summarized by three lines of thinking. First are research studies that look for misspecification of the simple linear two-factor model. Some, like Fama and French [1992] or Roll and Ross [1980] suggest a multifactor model. Others look for errors in the execution and design of the empirical tests (e.g., the survivorship bias problem discussed in Kothari, Shanken, and Sloan [1995]), frictions in capital markets (Amihud and Mendelsohn [1986]), or simply irrational behavior (Lakonishok, Shleifer, and Vishny [1994]). Finally is the possibility that the market risk premium and betas change over time (e.g., papers by Jagannathan and Wang [1996], Scruggs [1998], and Ferson and

[12] Selection bias is found in Compustat and results in upward-biased returns for high book/market equity portfolios for several reasons: In 1978 Compustat expanded its database from 2,700 to 6,000 companies, adding five years of data going back to 1973 with the likely effect that high book-to-market companies (in 1973) that did poorly or failed were unlikely to be found in the 1978 database, and vice versa. Kothari, Shanken, and Sloan [1995] found that the returns for the set of companies that were included on the CRSP data but not on Compustat were in fact lower than average.

Harvey [1999]). Gradual progress is being made, and most scholars believe that the notion of systematic risk is intuitive and important, but much remains to be done.

Berk, Green, and Naik [1999] present a dynamic theory of the firm that provides some intuition behind the empirical results of Fama and French [1996]. Each company is viewed as creating value through its stewardship of assets in place and discovery of valuable growth options. Good news about profitable growth drives up the share price but is also associated with relatively lower betas. Berk et al. describe the book-to-market ratio as a state variable that summarizes a firm's risk relative to the scale of its asset base. Strong profitable growth drives the book-to-market ratio down, drives beta down, and drives the required return down. Hence we would expect high book-to-market stocks to have lower cross-sectional returns than low book-to-market stocks—consistent with the empirical results. Also, in Berk et al., market capitalization is described as a state variable that captures the relative importance of existing assets versus growth options. Small stocks have higher expected returns than large because expected growth options are riskier and have a greater weighting in the present value of the company.

I. The Market Risk Premium

One of the most important numbers in financial economics is the market risk premium. If we are using a CAPM framework, it is the difference between the expected rate of return on the market portfolio and the expected return on the minimum-variance zero-beta portfolio (usually assumed to be the same as the risk-free rate). The market risk premium is a benchmark for required returns of index funds (and other mutual funds), and for the required returns on corporate capital investments. Practitioners are constantly confronted with how to estimate it. This turns out to be a thorny problem, interwoven with empirical tests of the CAPM as discussed in Section H.

The definition of the market risk premium, λ_1, is deceptively simple:

$$\lambda_1 = E(R_m) - E(R_z). \tag{6.41}$$

But over what interval of time should expectations be measured, how do we interpret ex post data, how do we account for sample selection biases, and should we use nominal or real rates of return? Not all of these questions are resolved. What follows is a discussion of some of the more important issues.

How do we interpret ex post data, when the definition of the risk premium is based on forward-looking expected returns on two portfolios, one with zero beta and the other with a beta of one? Normally, one assumes that the ex ante and ex post betas are identical, and proceeds to calculate arithmetic rather than geometric returns. To illustrate why, consider the following simple example. You invest \$100 at time zero, and there is a 50% chance that it can rise to \$200 or fall to \$50 in one year. After two years, it can rise to \$400, return to \$100, or fall to \$25. This event tree is illustrated in Fig. 6.9.

The usual calculation of the realized rate of return is the ex post geometric rate, defined as

$$R_p = [\Pi(1 + r_{pt})]^{1/T} - 1. \tag{6.42}$$

Note that there are no probabilities involved, because ex post the portfolio return has only one realization. For example, suppose that it followed the middle path in Fig. 6.9, where the stock price starts at \$100, climbs to \$200, then falls to \$100 at the end. The geometric average rate of

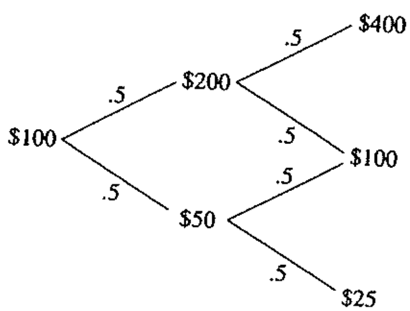


Figure 6.9 Example used to contrast arithmetic and geometric rates of return.

return along this path is 0%:

$$R_p = (1 + 100\%)(1 - 50\%) - 1 = (2)(1/2) - 1 = 0\%. \tag{6.43}$$

When attempting to evaluate the event tree of Fig. 6.9, ex ante, the problem is different because all branches are possible. Therefore we have to probability weight the outcomes of each branch in order to obtain the ex ante expected return, as an arithmetic average:

$$E(R_p) = \left[\sum p_i(1 + r_{pi})\right]^{1/T} - 1$$

$$\begin{aligned} E(R_p) &= [.25(400/100) + 2(.25)(100/100) + .25(25/100)]^{1/2} - 1 \\ &= 1.5625^{1/2} - 1 \\ &= 25\% \text{ per year.} \end{aligned}$$

Thus, while it is appropriate to use geometric averages when measuring historical portfolio performance, the asset pricing models are forward looking and therefore are based on arithmetic averages that are representative of expected returns.

The arithmetic average of the difference between the annual return on a value-weighted index of U.S. equities and returns on short-term U.S. Treasury bills has been 7.7 percent over the period 1926–1999 according to Ibbotson [2000]. But as Brown, Goetzmann, and Ross [1995] point out, this statistic includes survivorship bias and is biased upward. The intuition is fairly straightforward. Over the time interval 1926–2001, the U.S. index has not only been continuously traded, but also has been perhaps the single most successful of all indices that might have been chosen for investment in 1926. But what is the probability that over the next 76 years the U.S. markets will experience the same rate of success? Because of survivorship bias, many benchmarks of portfolio performance are upward biased, especially those whose weightings are chosen ex post—such as the Dow 30 Industrials, and the Standard & Poor's 500. No company in either index has ever been delisted or gone into Chapter 11. One estimate of the survivorship bias in the U.S. equity market index is 2%.

Another issue is how to define the risk-free rate. If we assume, for the time being, that interest rates are a fundamental factor that is not wholly captured in the CAPM, then portfolios with different interest rate sensitivity may have expected returns that differ as a function of their interest rate sensitivity. In this case, it is better to choose a risk-free rate that is zero default risk, namely, the rate on a U.S. government obligation, but with approximately the same sensitivity to interest rates

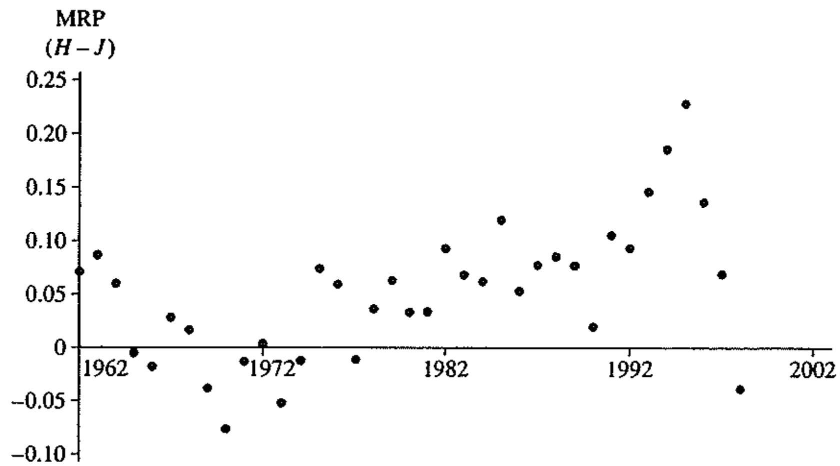


Figure 6.10 The market risk premium 1963–2002. *Source:* Monitor Group analysis.

as the equity (or market) index. Figure 6.10 shows the annual arithmetic market risk premium for an index of large company stocks minus the rate of return on an index of U.S. government bonds of intermediate maturity (which should have closer sensitivity to interest rates than an index based on short-term Treasury bills). If we were to use this data (1963–2002), our estimate of the market risk premium would be 11.9% (the average arithmetic return on the S & P 500 index) minus 7% (the average arithmetic return on intermediate-term U.S. government bonds). Thus, our estimate of the market risk premium would be roughly 5%, in nominal terms. We would obtain the same estimate in real terms if the expected inflation in our estimate of the risk-free rate is the same as the expected inflation in the index of large company stocks.

A time-series regression of the *market risk premium, MRP*, provides the following results (40 observations):

$$MRP = .041 + .0002\,(year)$$
$$(0.77) \quad (0.44) \tag{6.44}$$

The t-statistics are given in parentheses. The slope is not statistically different from zero. Consequently, there is absolutely no evidence of a trend in the market risk premium over time. Were the regression run on data from 1926 to 2002, the conclusion would be the same.

Fama and French [2002] estimate the market risk premium differently and use a longer time period, 1872–2000. Their logic is that the average stock return is the average dividend yield plus the average rate of capital gain:

$$A(R_t) = A(D_t/P_{t-1}) + A(GP_t). \tag{6.45}$$

If the dividend price ratio is stationary over time, then the rate of dividend growth approaches the compound rate of capital gain; therefore the annual growth in dividends can be substituted in place of the annual capital gain (in the equation above). Fama and French use this dividend growth model (and an earnings growth model) to estimate the unconditional expected stock return for the ith stock. Thus, they do not use ex post historical realizations of the difference between the return on the market portfolio and the risk-free rate. Instead they model ex ante differences between the expected return on the market portfolio and the risk-free rate. The simple arithmetic average for the period 1872–1950 is 4.40%, and the estimate from their dividend growth model is about the same, namely, 4.17%. However, for the period 1950–2000 the difference is large—7.43 versus 2.55%. There are several issues, however, for example, the growing use of stock repurchases and the declining use of dividends to deliver value to shareholders.

J. The Empirical Market Line

The empirical evidence has led scholars to conclude that the pure theoretical form of the CAPM does not agree well with reality. However, the empirical form of the model, which has come to be known as the *empirical market line*,

$$R_{it} = \widehat{\gamma}_{0t} + \widehat{\gamma}_{1t}\beta_{it} + \varepsilon_{it}, \tag{6.46}$$

does provide an adequate model of security returns. Obviously, if one can estimate a security's beta for a given time period, then by knowing the empirical market line parameters, one can estimate the security's required rate of return from Eq. (6.46).

Some (e.g., Ibbotson) have gone so far as to employ a version that does not use beta. For example, the expected rate of return on the equity of a company is argued to be a log-linear function of the size of the company and its book-to-market ratio:

$$R_{it} = \gamma_{0t} + \gamma_{1t}\ln(Size)_i + \gamma_{2t}(Book/Market)_i + \varepsilon_{it}. \tag{6.47}$$

Obvious questions facing a practitioner who attempts to use this version are "How do I apply it to projects that are obviously and significantly different in their total and systematic risk, although they have the same size?" or "What are the implications for the merger of two large companies?"

K. The Problem of Measuring Performance: Roll's Critique

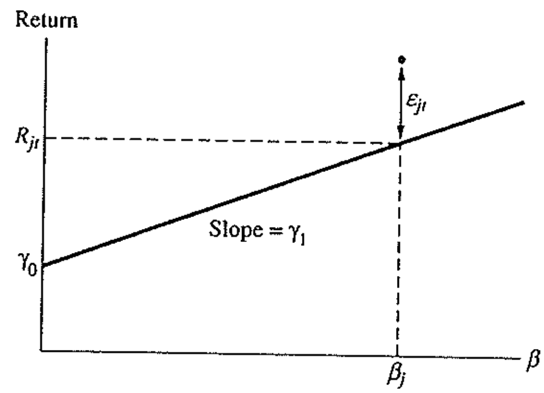
One of the potentially most useful applications of the securities market line in its ex post form, Eq. (6.35), or the empirical market line, Eq. (6.46), is that they might be used as benchmarks for security performance. The residual term, ε_{jt}, has been interpreted as abnormal because, as shown in Fig. 6.11, it represents return in excess of what is predicted by the security market line.

Roll [1977] takes exception to this interpretation of cross-section abnormal performance measures and to empirical tests of the CAPM in general. In brief, his major conclusions are

1. The only legitimate test of the CAPM is whether or not the market portfolio (which includes *all* assets) is mean-variance efficient.
2. If performance is measured relative to an index that is ex post efficient, then from the mathematics of the efficient set no security will have abnormal performance when measured as a departure from the security market line.[13]

[13] It is important to note that Roll does not take exception to time-series measures of performance such as those described by the market model in Chapter 11.

Figure 6.11 Abnormal return.



3. If performance is measured relative to an ex post inefficient index, then any ranking of portfolio performance is possible, depending on which inefficient index has been chosen.

This is a startling statement. It implies that even if markets are efficient and the CAPM is valid, then the cross-section security market line cannot be used as a means of measuring the ex post performance of portfolio selection techniques. Furthermore, the efficiency of the market portfolio and the validity of the CAPM are joint hypotheses that are almost impossible to test because of the difficulty of measuring the true market portfolio.

To understand Roll's critique, we must go back to the derivation of the zero-beta portfolio. Recall that if there is no risk-free asset, it is still possible to write the security market line as a combination of the market portfolio and a zero-beta portfolio that is uncorrelated with the market index. Therefore the expected return on any asset could be written as a two-factor model:

$$E(R_i) = E\left(R_z\right) + [E(R_m) - E(R_z)]\beta_i. \tag{6.48}$$

Roll points out that there is nothing unique about the market portfolio. It is always possible to choose any efficient portfolio as an index, then find the minimum-variance portfolio that is uncorrelated with the selected efficient index. This is shown in Fig. 6.12. Once this has been done, then Eq. (6.37) can be derived and written as

$$E(R_i) = E(R_{z,I}) + \left[E(R_I) - E(R_{z,I})\right]\beta_{i,I}. \tag{6.49}$$

Note that the market portfolio, R_m, has been replaced by any efficient index, R_I, and the beta is measured relative to the selected efficient index, $\beta_{i,I}$. Also, the zero-beta portfolio is measured relative to the index $R_{z,I}$. Because the expected return on any asset can be written as a linear function of its beta measured relative to any efficient index, it is not necessary to know the market index. One only need know the composition of an efficient index in order to write Eq. (6.49). Furthermore, if the index turns out to be ex post efficient, then every asset will fall exactly on the security market line. There will be no abnormal returns. If there are systematic abnormal returns, it simply means that the index that has been chosen is not ex post efficient.

The Roll critique does not imply that the CAPM is an invalid theory. However, it does mean that tests of the CAPM must be interpreted with great caution. The fact that portfolio residuals exhibited no significant departures from linearity merely implies that the market index that was selected

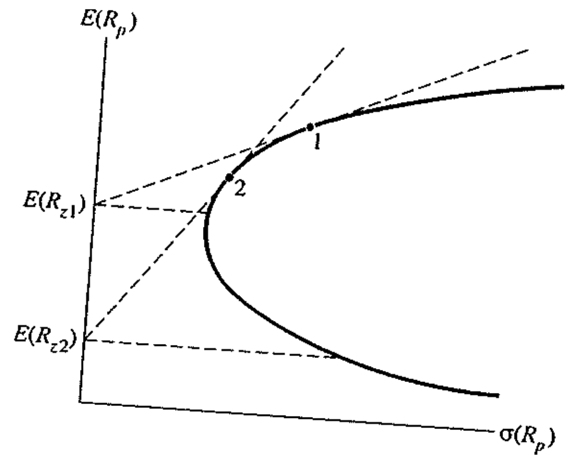


Figure 6.12 Two index portfolios with their respective orthogonal portfolios.

(usually an equally weighted index of all listed shares of common stock) was ex post efficient. In fact, the only way to test the CAPM directly is to see whether or not the true market portfolio is ex post efficient. Unfortunately, because the market portfolio contains all assets (marketable and nonmarketable, e.g., human capital, coins, houses, bonds, stocks, options, land, etc.), it is impossible to observe.

L. The Arbitrage Pricing Theory

1. The Theory

Formulated by Ross [1976], the *arbitrage pricing theory* (APT) offers a testable alternative to the capital asset pricing model. The CAPM predicts that security rates of return will be linearly related to a single common factor—the rate of return on the market portfolio. The APT is based on similar intuition but is much more general. It assumes that the rate of return on any security is a linear function of k factors:

$$\tilde{R}_i = E(\tilde{R}_i) + b_{i1}\tilde{F}_1 + \cdots + b_{ik}\tilde{F}_k + \tilde{\varepsilon}_i, \tag{6.50}$$

where

$\tilde{R}_i$ = the random rate of return on the ith asset,

$E(\tilde{R}_i)$ = the expected rate of return on the ith asset,

b_{ik} = the sensitivity of the ith asset's returns to the kth factor,

$\tilde{F}_k$ = the mean zero kth factor common to the returns of all assets,

$\tilde{\varepsilon}_i$ = a random zero mean noise term for the ith asset.

As we shall see later on, the CAPM may be viewed as a special case of the APT when the market rate of return is assumed to be the single relevant factor.

The APT is derived under the usual assumptions of perfectly competitive and frictionless capital markets. Furthermore, individuals are assumed to have homogeneous beliefs that the random returns for the set of assets being considered are governed by the linear k-factor model given in Eq. (6.50). The theory requires that the number of assets under consideration, n, be much larger than the number of factors, k, and that the noise term, $\tilde{\varepsilon}_i$, be the unsystematic risk component for the ith asset. It must be independent of all factors and all error terms for other assets.

The most important feature of the APT is reasonable and straightforward. In equilibrium all portfolios that can be selected from among the set of assets under consideration and that satisfy the conditions of (1) using no wealth and (2) having no risk must earn no return on average. These portfolios are called *arbitrage portfolios*. To see how they can be constructed, let w_i be the *change* in the dollar amount invested in the ith asset as a percentage of an individual's total invested wealth. To form an arbitrage portfolio that requires no change in wealth (or is said to be self-financing), the usual course of action would be to sell some assets and use the proceeds to buy others. Mathematically, the zero change in wealth is written as

$$\sum_{i=1}^{n} w_i = 0. \tag{6.51}$$

If there are n assets in the arbitrage portfolio, then the additional portfolio return gained is

$$\begin{aligned}\tilde{R}_p &= \sum_{i=1}^{n} w_i \tilde{R}_i \\ &= \sum_i w_i E(\tilde{R}_i) + \sum_i w_i b_{i1} \tilde{F}_1 + \cdots + \sum_i w_i b_{ik} \tilde{F}_k + \sum_i w_i \tilde{\varepsilon}_i. \end{aligned} \tag{6.52}$$

To obtain a riskless arbitrage portfolio it is necessary to eliminate both diversifiable (i.e., unsystematic or idiosyncratic) and undiversifiable (i.e., systematic) risk. This can be done by meeting three conditions: (1) selecting percentage changes in investment ratios, w_i, that are small, (2) diversifying across a large number of assets, and (3) choosing changes, w_i, so that for each factor, k, the weighted sum of the systematic risk components, b_k, is zero. Mathematically, these conditions are

$$w_i \approx 1/n, \tag{6.53a}$$

$$n \text{ chosen to be a large number,} \tag{6.53b}$$

$$\sum_i w_i b_{ik} = 0 \quad \text{for each factor.} \tag{6.53c}$$

Because the error terms, ε_i, are independent, the law of large numbers guarantees that a weighted average of many of them will approach zero in the limit as n becomes large. In other words, costless diversification eliminates the last term (the unsystematic or idiosyncratic risk) in Eq. (6.50). Thus, we are left with

$$\tilde{R}_p = \sum_i w_i E\left(\tilde{R}_i\right) + \sum_i w_i b_{i1} \tilde{F}_1 + \cdots + \sum_i w_i b_{ik} \tilde{F}_k. \tag{6.54}$$

At first glance the return on our portfolio appears to be a random variable, but we have chosen the weighted average of the systematic risk components for each factor to be equal to zero ($\sum w_i b_{ik} = 0$). This eliminates all systematic risk. One might say that we have selected an arbitrage portfolio with zero beta in each factor. Consequently, the return on our arbitrage portfolio becomes a constant. Correct choice of the weights has eliminated all uncertainty, so that R_p is not a random variable. Therefore Eq. (6.52) becomes

$$R_p = \sum_i w_i E(\widetilde{R}_i). \tag{6.55}$$

Recall that the arbitrage portfolio, so constructed, has no risk (of any kind) and requires no new wealth. If the return on the arbitrage portfolio were not zero, then it would be possible to achieve an infinite rate of return with no capital requirements and no risk. Such an opportunity is clearly impossible if the market is to be in equilibrium. In fact, if the individual arbitrageur is in equilibrium (hence content with his or her current portfolio), then the return on any and all arbitrage portfolios must be zero. In other words,

$$R_p = \sum_i w_i E(\widetilde{R}_i) = 0. \tag{6.56}$$

Eqs. (6.51), (6.53c), and (6.56) are really statements in linear algebra. Any vector that is orthogonal to the constant vector, that is,[14]

$$\sum_{i=1}^{n} (w_i) \cdot \mathbf{e} = 0,$$

and to each of the coefficient vectors, that is,

$$\sum_i w_i b_{ik} = 0 \quad \text{for each } k,$$

must also be orthogonal to the vector of expected returns, that is,

$$\sum_i w_i E(\widetilde{R}_i) = 0.$$

An algebraic consequence of this statement is that the expected return vector must be a linear combination of the constant vector and the coefficient vectors. Algebraically, there must exist a set of $k + 1$ coefficients, $\lambda_0, \lambda_1, \ldots, \lambda_k$ such that

$$E(\widetilde{R}_i) = \lambda_0 + \lambda_1 b_{i1} + \cdots + \lambda_k b_{ik}. \tag{6.57}$$

Recall that the b_{ik} are the "sensitivities" of the returns on the ith security to the kth factor. If there is a riskless asset with a riskless rate of return, R_f, then $b_{0k} = 0$ and

$$R_f = \lambda_0.$$

Hence Eq. (6.45) can be rewritten in "excess returns form" as

$$E(R_i) - R_f = \lambda_1 b_{i1} + \cdots + \lambda_k b_{ik}. \tag{6.58}$$

[14] Note that Eq. (6.40) says that the sum of the investment weights equals zero. This is really a no-wealth constraint. No new wealth is required to take an arbitrage position. Recall that $\mathbf{e}$ is a column vector of ones.

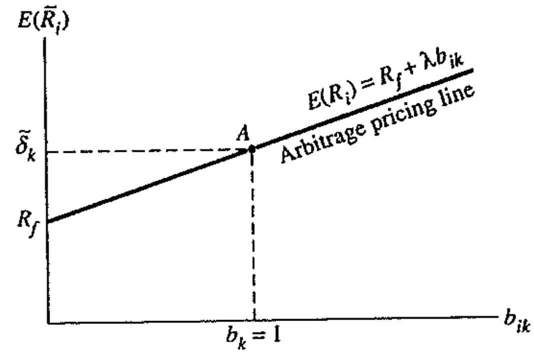


Figure 6.13 The arbitrage pricing line.

Figure 6.13 illustrates the arbitrage pricing relationship (6.58) assuming that there is only a single stochastic factor, k. In equilibrium, all assets must fall on the *arbitrage pricing line*. A natural interpretation for λ_k is that it represents the risk premium (i.e., the price of risk), in equilibrium, for the kth factor. Because the arbitrage pricing relationship is linear, we can use the slope-intercept definition of a straight line to rewrite Eq. (6.58) as

$$E(R_i) = R_f + \left[\overline{\delta}_k - R_f\right] b_{ik},$$

where $\overline{\delta}_k$ is the expected return on a portfolio with unit sensitivity to the kth factor and zero sensitivity to all other factors. Therefore the risk premium, λ_k, is equal to the difference between (1) the expectation of a portfolio that has unit response to the kth factor and zero response to the other factors and (2) the risk-free rate, R_f:

$$\lambda_k = \overline{\delta}_k - R_f.$$

In general the arbitrage pricing theory can be rewritten as

$$E(R_i) - R_f = \left[\overline{\delta}_1 - R_f\right] b_{i1} + \cdots + \left[\overline{\delta}_k - R_f\right] b_{ik}. \tag{6.59}$$

If Eq. (6.59) is interpreted as a linear regression equation (assuming that the vectors of returns have a joint normal distribution and that the factors have been linearly transformed so that their transformed vectors are orthonormal), then the coefficients, b_{ik}, are defined in exactly the same way as beta in the capital asset pricing model, that is,

$$b_{ik} = \frac{\text{COV}(R_i, \delta_k)}{\text{VAR}(\delta_k)} \tag{6.60}$$

where

$\text{COV}(R_i, \delta_k)$ = the covariance between the ith asset's returns and the linear transformation of the kth factor,

$\text{VAR}(\delta_k)$ = the variance of the linear transformation of the kth factor.

Hence the CAPM is seen to be a special case of the APT (where asset returns are assumed to be joint normal).

The arbitrage pricing theory is much more robust than the capital asset pricing model for several reasons:

1. The APT makes no assumptions about the empirical distribution of asset returns.
2. The APT makes no strong assumptions about individuals' utility functions (at least nothing stronger than greed and risk aversion).
3. The APT allows the equilibrium returns of assets to be dependent on many factors, not just one (e.g., beta).
4. The APT yields a statement about the relative pricing of any subset of assets; hence one need not measure the entire universe of assets in order to test the theory.
5. There is no special role for the market portfolio in the APT, whereas the CAPM requires that the market portfolio be efficient.
6. The APT is easily extended to a multiperiod framework (see Ross [1976]).

Suppose that asset returns are determined by two underlying factors such as unanticipated changes in real output and unanticipated inflation. The arbitrage pricing theory can easily account for the effect of changes in both factors on asset returns. Because the capital asset pricing model relies on a single factor (the market index), it cannot do as well. Using the CAPM is a little like being lost in the clouds while piloting a private plane. You call the air controller and ask, "Where am I?" If the controller is using a unidimensional model like the CAPM, he or she is likely to respond, "Two hundred miles from New York City." Obviously, this is not a very helpful answer. A multidimensional model like the APT would be more useful. It would be nice to know latitude, longitude, and altitude.

Figure 6.14 illustrates the same point. The factor loadings (or factor sensitivities), b_{i1} and b_{i2}, for our two hypothetical factors—changes in unanticipated real output and changes in unanticipated inflation—are plotted on the axes. The origin represents the risk-free rate that is the rate of return received when an asset has zero beta in both factors. Points along the diagonal dashed lines have equal expected return but not the same risk. For example, all points along the line OJ have an expected rate of return equal to the risk-free rate but are not riskless portfolios. If the risk-free rate is 10%, one can obtain that rate either with a truly riskless portfolio that pays 10% in every state of nature or with a second portfolio that has positive sensitivity to one factor and negative sensitivity to the other factor.

Suppose the arbitrage pricing model, Eq. (6.59),

$$E(R_i) - R_f = \left[\bar{\delta}_1 - R_f\right] b_{i1} + \left[\bar{\delta}_2 - R_f\right] b_{i2},$$

is estimated to have the following numerical values: $R_f = 10\%$, $\bar{\delta}_1 = 20\%$, and $\bar{\delta}_2 = 15\%$. If b_{i1} is plotted on the vertical axis, then the vertical intercept for a given return $E(R_i)$ is

$$\alpha = \text{vertical intercept} = \frac{E(R_i) - R_f}{\bar{\delta}_1 - R_f},$$

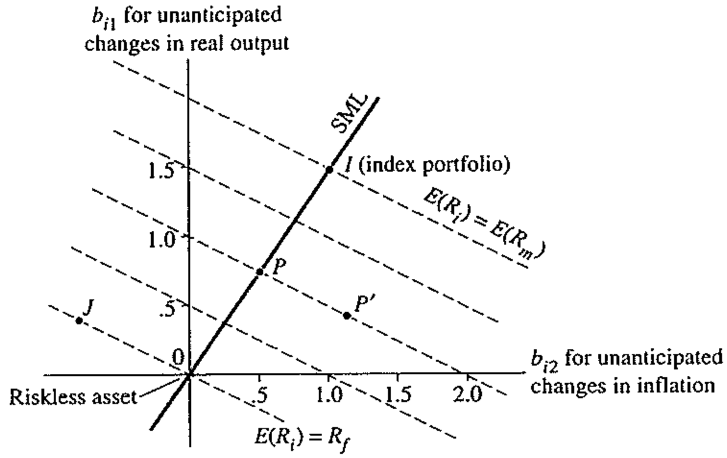


Figure 6.14 A graph of the CAPM and the APT.

and the slope of the equal return line is

$$m = \text{slope} = -\frac{\bar{\delta}_2 - R_f}{\bar{\delta}_1 - R_f},$$

where the equation for the return line is $b_{i1} = \alpha + mb_{i2}$.

Next, suppose that we know that a CAPM efficient index portfolio has been chosen, that its expected return is 30%, and that its sensitivity to changes in unanticipated real output and changes in unanticipated inflation are $b_{i1} = 1.5$ and $b_{i2} = 1.0$. The CAPM index portfolio is plotted at point I in Fig. 6.14. We know from the CAPM, Eq. (6.9), that the security market line is represented by all linear combinations of the risk-free asset and the market index portfolio. Therefore the security market line is represented by the ray OI from the origin (which is the risk-free asset) to the efficient index portfolio (point I).

The CAPM measures risk only in one dimension, rather than two. If we are told that the portfolio's CAPM β is .5, then it will plot at point P in Fig. 6.14, halfway between the riskless asset and the index portfolio. However, according to the APT there are an infinite number of portfolios, all with the same expected return as portfolio P and each having a different sensitivity to the APT risk parameters b_{i1} and b_{i2}. If people are in fact sensitive to more than one type of risk when choosing among portfolios of equal return, then the APT is superior to the CAPM because the CAPM is unidimensional in risk. It is perfectly reasonable that portfolio P' might be preferred to portfolio P by some investors because it has the same return as portfolio P but a preferable combination of sensitivities to the underlying factors. For example, a public pension fund manager might not care much about the sensitivity of the value of the fund to industrial production but might be very concerned about hedging against unexpected changes in inflation. Later on we shall discuss some empirical work that provides evidence that more than one factor is significant in explaining security returns.

2. A Numerical Example

To illustrate how arbitrage pricing might be employed in practice, suppose that empirical work reveals that the expected returns on assets can be explained by two factors, F_1 and F_2. Table 6.6

Table 6.6 Data for an APT Example

State of Nature	Prob.	Asset Returns (%) $\tilde{X}$	$\tilde{Y}$	$\tilde{Z}$	Transformed Factor Changes (%) $\bar{\delta}_1$	$\bar{\delta}_2$
Horrid	.2	−55.23	623.99	53.00	−10.00	−5.00
Bad	.2	70.70	10.00	413.37	−5.00	38.48
Average	.2	−9.00	25.00	−1493.12	25.00	8.00
Good	.2	−12.47	−3771.42	1058.75	40.00	−1.44
Great	.2	61.00	3237.44	83.00	50.00	0.00

shows the subjectively estimated returns on three assets (X, Y, and Z) and the changes in (orthogonal transformations of) the two factors for five equally likely states of nature. To make the problem tractable, assume that all factors and returns are normally distributed. In addition, suppose we know that the expected return on the risk-free asset, R_f, is 10%.

One of the requirements of the factor analysis program usually used to test the arbitrage pricing model is that we are working with linear transformations of the underlying factors. The transformations must be orthogonal (i.e., the product of their row and column vectors must equal zero). This is shown below for the transformed factors in Table 6.6:

$$[-10 \quad -5 \quad 25 \quad 40 \quad 50] \begin{bmatrix} -5.00 \\ 38.48 \\ 8.00 \\ -1.44 \\ 0 \end{bmatrix} = 0.$$

How can we tell from the bewildering set of numbers in Table 6.6 whether or not there are any arbitrage opportunities? And if there are, how can we take advantage of them by forming an arbitrage portfolio?

If there are only two factors that govern all returns, then the APT becomes

$$E(R_i) = R_f + \left[\bar{\delta}_1 - R_f\right] b_{i1} + \left[\bar{\delta}_2 - R_f\right] b_{i2}.$$

The data from Table 6.6 can be used to compute all the terms on the right-hand side of this equation. The factor loadings (or sensitivities) are the same as beta, Eq. (6.60), given the assumption of normally distributed returns and orthogonal transformations of the factors. Using asset X as an example, we need to calculate

$$b_{x1} = \frac{\text{COV}(X, \delta_1)}{\text{VAR}(\delta_1)} = \frac{285.0}{570.0} = .5.$$

The computations are done in Table 6.7. Given that $b_{x1} = .5$, we know that a 1% increase in factor 1 will result in a .5% increase in the return on security X. We can think of the factor loadings (or sensitivities) in exactly the same way as we thought of beta (systematic risk) in the CAPM. The expectations of each asset and transformed factor and the factor loadings (sensitivities) are given in Table 6.8. By substituting these data into the APT equation, we can determine the market

Table 6.7 Calculating b_{x1} from the Data in Table 6.6

$p_i X_i$		$p_i \delta_{1i}$		$p_i(\delta_{1i} - \overline{\delta}_1)^2$	
$.2(-55.23) =$	-11.046	$.2(-10) =$	-2.0	$.2(-10-20)^2 =$	180
$.2(70.70) =$	14.140	$.2(-5) =$	-1.0	$.2(-5-20)^2 =$	125
$.2(-9.00) =$	-1.800	$.2(25) =$	5.0	$.2(25-20)^2 =$	5
$.2(-12.47) =$	-2.494	$.2(40) =$	8.0	$.2(40-20)^2 =$	80
$.2(-61.00) =$	12.200	$.2(50) =$	10.0	$.2(50-20)^2 =$	180
$\overline{X} =$	11.000	$\overline{\delta}_1 =$	20.0	$\text{VAR}(\delta_1) =$	570

$p_i(X_i - \overline{X})(\delta - \overline{\delta}_1)$	
$.2(-66.23)(-30) =$	397.38
$.2(59.70)(-25) =$	-298.50
$.2(-20.00)(5) =$	-20.00
$.2(-23.47)(20) =$	-93.98
$.2(50.00)(30) =$	300.00
$\text{COV}(X, \delta_1) =$	285.00

$$b_{x1} = \frac{285.00}{570.00} = .5$$

Table 6.8 Statistics Computed from Table 6.6

		Factor Loadings		
Asset	R_1	b_{i1}	b_{i2}	**Transformed Factor Expectations**
X	11%	.5	2.0	$\overline{\delta}_1 = 20\%$
Y	25	1.0	1.5	$\overline{\delta}_2 = 8\%$
Z	23	1.5	1.0	

equilibrium rate of return, $E(R_i)$, for each of the three assets. This is done below:

$$E(R_x) = .10 + [.20 - .10]\,0.5 + [.08 - .10]\,2.0 = 11\%,$$

$$E(R_y) = .10 + [.20 - .10]\,1.0 + [.08 - .10]\,1.5 = 17\%,$$

$$E(R_z) = .10 + [.20 - .10]\,1.5 + [.08 - .10]\,1.0 = 23\%.$$

Note that the equilibrium return, $E(R_i)$, on assets X and Z is the same as the projected return, $\overline{R}_i$, computed from the data. Hence no arbitrage opportunities exist for trading in these two assets. On the other hand the projected return on asset Y, $\overline{R}_y$, is 25% when computed from the data, and the market equilibrium return, $E(R_y)$, is only 17%. Therefore by selling the correct proportions of assets X and Z and buying asset Y, we can form an arbitrage portfolio that requires no new capital, has no change in risk, and earns a positive rate of return.

Suppose that we currently have one third of our wealth in each of the three assets. How should we change our portfolio to form a riskless arbitrage position? It turns out that so long as there are more assets than factors, there are virtually an infinite number of ways of forming arbitrage portfolios. But let us suppose that we desire to put the maximum investment into asset Y without actually selling short either X or Z. Our investment proportion in Y would go from one third to one; thus the change in holdings of asset Y would be $w_y = \frac{2}{3}$. We also require that the change portfolio have zero beta in each factor and that it need no net wealth. These conditions are stated in Eqs. (6.51) and (6.53c), which are repeated below:

$$\left(\sum_{i=1}^{3} w_i\right) \cdot \mathbf{e} = \mathbf{0}.$$

$$\sum_{i=1}^{3} w_i b_{ik} = 0 \quad \text{for each } k.$$

Expanding these equations, we have

$$w_x + w_y + w_z = 0,$$

$$w_x b_{x1} + w_y b_{y1} + w_z b_{z1} = 0 \quad \text{for factor 1,}$$

$$w_x b_{x2} + w_y b_{y2} + w_z b_{z2} = 0 \quad \text{for factor 2.}$$

And substituting in the numbers of our problem, we get

$$w_x + \frac{2}{3} + w_z = 0,$$

$$w_x(.5) + \frac{2}{3}(1.0) + w_z(1.5) = 0,$$

$$w_x(2.0) + \frac{2}{3}(1.5) + w_z(1.0) = 0.$$

Solving, we find that

$$w_x = -\frac{1}{3}, \qquad w_y = \frac{2}{3}, \qquad w_z = -\frac{1}{3}.$$

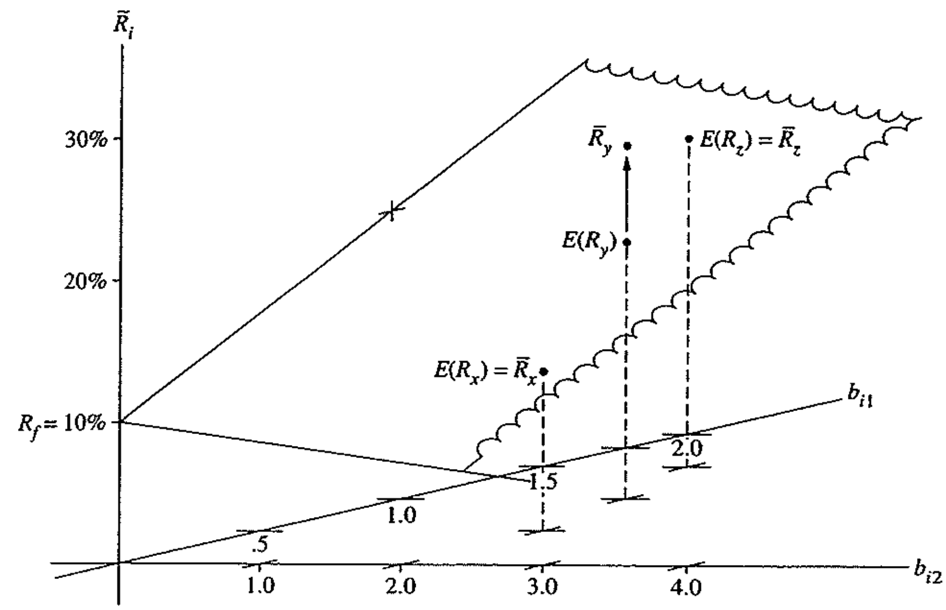
Thus we would sell all our holdings in assets X and Z and invest the proceeds in asset Y.

This strategy would require no new wealth and would have no change in risk. Note that our risk position before making the change was

$$\frac{1}{3}(.5) + \frac{1}{3}(1.0) + 0(1.5) = 1.0 \quad \text{for factor 1,}$$

$$\frac{1}{3}(2) + \frac{1}{3}(1.5) + \frac{1}{3}(1.0) = 1.5 \quad \text{for factor 2.}$$

Figure 6.15 Arbitrage pricing plane for two factors.



Since our systematic risk has not changed, the extra systematic risk created by the arbitrage portfolio is zero.[15] Our originally projected portfolio return was

$$\frac{1}{3}(11\%) + \frac{1}{3}(25\%) + \frac{1}{3}(23\%) = 19.67\%.$$

But after investing in the arbitrage portfolio, we project a rate of return of

$$0(11\%) + 1(25\%) + 0(23\%) = 25\%.$$

Thus the arbitrage portfolio increases our return by 5.33% without changing our systematic risk.

Figure 6.15 provides a visual description of the example problem. Expected rates of return are plotted on the vertical axis, and asset betas in each of the two factors are plotted along the horizontal axes. Note that the expected returns for assets X and Z plot exactly on the arbitrage pricing plane. They are in equilibrium. But asset Y plots above the plane. Its return lies considerably above what the market requires for its factor loadings, b_{y1} and b_{y2}. Hence an arbitrage opportunity exists. If enough people take advantage of it, the price of asset Y will rise, thereby forcing its rate of return down and back into equilibrium.

[15] Because there is idiosyncratic risk to contend with, total risk will in fact change. However, with well-diversified arbitrage portfolios this problem vanishes because diversification reduces idiosyncratic risk until it is negligible.

M. Empirical Tests of the Arbitrage Pricing Theory

Papers by Gehr [1975], Roll and Ross [1980], Reinganum [1981a], Chen [1983], Chen, Roll, and Ross [1986], and Conner and Korajczyk [1993] have tested the APT using data on equity rates of return for New York and American Stock Exchange listed stocks. The usual empirical procedure has the following steps:

1. Collect a time series of stock returns data for a group of stocks.
2. Compute the empirical variance-covariance matrix from the returns data.
3. Use a (maximum-likelihood) factor analysis procedure to identify the number of factors and their factor loadings, b_{ik}.
4. Use the estimated factor loadings, b_{ik}, to explain the cross-sectional variation of individual estimated expected returns and to measure the size and statistical significance of the estimated risk premia associated with each factor.

The Roll and Ross [1980] study used daily returns data for NYSE and AMEX companies listed on the exchanges on both July 3, 1962, and December 31, 1972. There were a maximum of 2,619 daily returns for each stock. The 1,260 securities selected were divided alphabetically into groups of 30. For each group of 30 the procedure described above was carried out. The analysis showed that there are at least three and probably four "priced" factors. There may be other zero-priced factors, but this procedure cannot identify them because their regression coefficients in step 4 would be zero.

One of the frustrating things about using factor analysis to test the APT is that this procedure cannot tell us what the factors are. However, we can reject the APT if a specified alternative variable such as the total variance of individual returns, firm size, or the asset's last period return were to be significant in explaining the expected returns. Roll and Ross [1980], after correcting for the problem that positive skewness in lognormal returns can create dependence between the sample mean and the sample standard deviation, found that the total variance of security returns does not add any explanatory power for estimated expected returns. Therefore the APT could not be rejected on this basis. Although a different procedure was employed by Chen [1983], he was able to confirm this result. He also found that the asset's last period return added no explanatory power.

Currently there is a question whether or not firm size can be used to refute the APT because it adds explanatory power to the factor loadings. Reinganum [1981a] finds that it does. His test consisted of estimating the factor loadings in year $(t-1)$ for all securities, then combining securities with similar loadings into control portfolios. In year t, excess security returns are computed by subtracting the daily control portfolio returns from the daily security returns. Finally, with excess returns in hand, the securities are ranked on the basis of the market value of all the firm's outstanding common stock at period $(t-1)$. The APT predicts (if factor loadings are stationary across time) that all deciles of the market value ranked excess returns should have the same mean. Reinganum finds that there are significant differences between the mean excess returns and rejects the APT. Chen [1983], on the other hand, finds that firm size adds no explanatory power. His procedure uses Reinganum's data for the market value of each firm's equity. He divides the sample of firms into two groups—those with greater than the median market value and those with less. Then portfolios are formed from the high- and low-value firms so that the following conditions are satisfied: (1) each security in the portfolio has nonnegative weight and the weight should not be too far from $1/n$, where n is the number of securities in the portfolio, and (2) the resultant two portfolios have exactly the same factor loadings (arbitrage risk factors) in each factor. The factor loadings are determined by using returns data from odd days during each test period; the even-days returns from the same test period are used for measuring the average portfolio returns of the high- and low-valued portfolios. If the APT is correct, the returns of the two portfolios should not be statistically different because they are selected to have the same "risk" as determined by the factor loadings. In only one of the four periods tested is the difference in returns statistically different at the 95% confidence level. Therefore Chen argues that firm size effects have insufficient explanatory power to reject the APT.

There is one parameter in the APT, namely the intercept term, $\lambda_0 = R_f$, that should be identical across groups of securities when the model is estimated separately for each group during a time period. Other factors need not be the same because the factor loadings are not unique from group to group. For example, factor 1 in group A might correspond to factor 3 in group B. The intercept term, λ_0, however, has the same meaning in each group because it is the return on an asset that has no sensitivity to the common factors. Roll and Ross [1980] tested for the equivalence of the $\hat{\lambda}_0$ terms across 38 groups and found absolutely no evidence that the intercept terms were different. Again, the APT could not be rejected.

A direct comparison of the APT and the CAPM was performed by Chen [1983]. First, the APT model was fitted to the data as in the following equation:

$$\tilde{R}_i = \hat{\lambda}_0 + \hat{\lambda}_1 b_{i1} + \cdots + \hat{\lambda}_n b_{in} + \hat{\varepsilon}_i. \qquad \text{(APT)}$$

Then the CAPM was fitted to the same data:

$$\tilde{R}_i = \hat{\lambda}_0 + \hat{\lambda}_1 \hat{\beta}_i + \tilde{\eta}_i. \qquad \text{(CAPM)}$$

Next the CAPM residuals, η_i, were regressed on the arbitrage factor loadings, $\hat{\lambda}_k$, and the APT residuals, ε_i, were regressed on the CAPM coefficients. The results showed that the APT could explain a statistically significant portion of the CAPM residual variance, but the CAPM could not explain the APT residuals. This is strong evidence that the APT is a more reasonable model for explaining the cross-sectional variance in asset returns.

Although it is mathematically impossible to use factor analysis to unambiguously identify the underlying factors that drive security returns, Chen, Roll, and Ross [1986] have correlated various macroeconomic variables with returns on five portfolios that mimic what the underlying factors might be. Four macroeconomic variables were significant:

1. An index of industrial production
2. Changes in a default risk premium (measured by the differences in promised yields to maturity on AAA versus Baa corporate bonds)
3. Twists in the yield curve (measured by differences in promised yields to maturity on long- and short-term government bonds)
4. Unanticipated inflation

The economic logic underlying these variables seems to make sense. Common stock prices are the present values of discounted cash flows. The industrial production index is obviously related to profitability. The remaining variables are related to the discount rate. Conner and Korajczyk [1993] find evidence supporting the existence of one to six factors in a data set consisting of monthly returns of NYSE and AMEX stocks from 1967 to 1991.

The intuition behind these factors is useful for portfolio management. For example, it has often been stated that common stocks are not a good hedge against inflation. Although it is true if one holds an equally weighted portfolio of all stocks, the logic of factor analysis suggests that there is a well-diversified subset of common stocks that is in fact a good hedge against inflation. Since the factors are mutually orthogonal, one can (at least in principle) choose a portfolio that is hedged against inflation risk without changing the portfolio sensitivity to any of the other three above-mentioned factors.

Summary

This chapter has derived two theoretical models, the CAPM and the APT, that enable us to price risky assets in equilibrium. Within the CAPM framework the appropriate measure of risk is the covariance of returns between the risky asset in question and the market portfolio of all assets. The APT model is more general. Many factors (not just the market portfolio) may explain asset returns. For each factor the appropriate measure of risk is the sensitivity of asset returns to changes in the factor. For normally distributed returns the sensitivity is analogous to the beta (or systematic risk) of the CAPM.

The CAPM was shown to provide a useful conceptual framework for capital budgeting and the cost of capital. It is also reasonably unchanged by the relaxation of many of the unrealistic assumptions that made its derivation simpler. Finally, although the model is not perfectly validated by empirical tests, its main implications are upheld—namely, that systematic risk (beta) is a valid measure of risk, that the model is linear, and that the trade-off between return and risk is positive.

The APT can also be applied to cost of capital and capital budgeting problems. The earliest empirical tests of the APT have shown that asset returns are explained by three or possibly more factors and have ruled out the variance of an asset's own returns as one of the factors.

PROBLEM SET

6.1 Let us assume a normal distribution of returns and risk-averse utility functions. Under what conditions will all investors demand the same portfolio of risky assets?

6.2 The following data have been developed for the Donovan Company, the manufacturer of an advanced line of adhesives:

State	Probability	Market Return, R_m	Return for the Firm, R_j
1	.1	−.15	−.30
2	.3	.05	.00
3	.4	.15	.20
4	.2	.20	.50

The risk-free rate is 6%. Calculate the following:

(a) The expected market return.

(b) The variance of the market return.

(c) The expected return for the Donovan Company.

(d) The covariance of the return for the Donovan Company with the market return.
(e) Write the equation of the security market line.
(f) What is the required return for the Donovan Company? How does this compare with its expected return?

6.3 The following data have been developed for the Milliken Company:

Year	Market Return	Company Returns
1978	.27	.25
1977	.12	.05
1976	−.03	−.05
1975	.12	.15
1974	−.03	−.10
1973	.27	.30

The yield to maturity on Treasury bills is .066 and is expected to remain at this point for the foreseeable future. Calculate the following:

(a) The expected market return.
(b) The variance of the market return.
(c) The expected rate of return for the Milliken Company.
(d) The covariance of the return for the Milliken Company with the return on the market.
(e) Write the equation of the security market line.
(f) What is the required return for the Milliken Company?

6.4 For the data in Table Q6.4 (page 190), perform the indicated calculations.

6.5 For the data in Table Q6.5 (page 191), calculate the items indicated.

6.6 What are the assumptions sufficient to guarantee that the market portfolio is an efficient portfolio?

6.7 In the CAPM is there any way to identify the investors who are more risk averse? Explain. How would your answer change if there were not a riskless asset?

6.8 Given risk-free borrowing and lending, efficient portfolios have no unsystematic risk. True or false? Explain.

6.9 What is the beta of an efficient portfolio with $E(R_j) = 20\%$ if $R_f = 5\%$, $E(R_m) = 15\%$, and $\sigma_m = 20\%$? What is its σ_j? What is its correlation with the market?

6.10 Given the facts of Problem 6.9, and that the common stock of the Rapid Rolling Corporation has $E(R_k) = 25\%$ and $\sigma_k^2 = 52\%$, what is the systematic risk of the common stock? What is its unsystematic risk?

6.11 (a) If the expected rate of return on the market portfolio is 14% and the risk-free rate is 6%, find the beta for a portfolio that has expected rate of return of 10%. What assumptions concerning this portfolio and/or market conditions do you need to make to calculate the portfolio's beta?
(b) What percentage of this portfolio must an individual put into the market portfolio in order to achieve an expected return of 10%?

Table Q6.4 Estimates of Market Parameters

Year	S&P 500 Index	Percentage Change in Price	Dividend Yield	Percentage Return	Return Deviation	Market Variance	
	P_t	$\frac{P_t}{P_{t-1}} - 1$	$\frac{Div_t}{P_t}$	R_{mt} (3 + 4)	$(R_{mt} - \overline{R}_m)$ (5 − R_m)	$(R_{mt} - \overline{R}_m)^2$ (6^2)	R_f
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1960	55.84						
1961	66.27		.0298				.03
1962	62.38		.0337				.03
1963	69.87		.0317				.03
1964	81.37		.0301				.04
1965	88.17		.0300				.04
1966	85.26		.0340				.04
1967	91.93		.0320				.05
1968	98.70		.0307				.05
1969	97.84		.0324				.07
1970	83.22		.0383				.06

a. $\overline{R}_m = ?$
b. $\text{VAR}(R_m) = ?$
c. $\sigma(R_m) = ?$

6.12 You believe that the Beta Alpha Watch Company will be worth \$100 per share one year from now. How much are you willing to pay for one share today if the risk-free rate is 8%, the expected rate of return on the market is 18%, and the company's beta is 2.0?

6.13 Given the following variance-covariance matrix and expected returns vector (for assets X and Y, respectively) for a two-asset world:

$$\sum = \begin{bmatrix} .01 & 0 \\ 0 & .0064 \end{bmatrix}, \quad \overline{R}_I' = [.2 \quad .1]$$

(a) What is the expected return of a zero-beta portfolio, given that 50% of the index portfolio is invested in asset X and asset Y?
(b) What is the vector of weights in the global minimum-variance portfolio?
(c) What is the covariance between the global minimum-variance portfolio and the zero-beta portfolio?
(d) What is the equation of the market line?

6.14 Given the following variance-covariance matrix, calculate the covariance between portfolio A, which has 10% in asset 1 and 90% in asset 2, and portfolio B, which has 60% in asset 1 and 40% in asset 2:

Table Q6.5 Calculations of Beta for General Motors

Year	GM Price	Percentage Change in Price	Dividend Yield	Percentage Return	Deviation of Returns	Variance of Returns	Covariance with Market
				R_{jt}	$(R_{jt} - \overline{R}_j)$	$(R_{jt} - \overline{R}_j)^2$	$(R_{jt} - \overline{R}_j)(R_{mt} - R_m)$
	P_t	$\frac{P_t}{P_{t-1}} - 1$	$\frac{Div_t}{P_t}$	$(3+4)$	$(5 - \overline{R}_j)$	(6^2)	**(Col. 6 × Q6.4 Col. 6)**
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
1960	48						
1961	49		.05				
1962	52		.06				
1963	74		.05				
1964	90		.05				
1965	102		.05				
1966	87		.05				
1967	78		.05				
1968	81		.05				
1969	74		.06				
1970	70		.05				

a. $\overline{R}_j = ?$
b. $\text{VAR}(R_j) = ?$
c. $\text{COV}(R_j, R_m) = ?$
d. $\beta_j = ?$

$$\sum = \begin{bmatrix} .01 & -.02 \\ -.02 & .04 \end{bmatrix}$$

6.15 Suppose that securities are priced as if they are traded in a two-parameter economy. You have forecast the correlation coefficient between the rate of return on Knowlode Mutual Fund and the market portfolio at .8. Your forecast of the standard deviations of the rates of return are .25 for Knowlode, and .20 for the market portfolio. How would you combine the Knowlode Fund and a riskless security to obtain a portfolio with a beta of 1.6?

6.16 You currently have 50% of your wealth in a risk-free asset and 50% in the four assets below:

Asset	Expected Return on Asset i (%)	β_i	Percentage Invested in Asset i (%)
$i = 1$	7.6	.2	10
$i = 2$	12.4	.8	10
$i = 3$	15.6	1.3	10
$i = 4$	18.8	1.6	20

If you want an expected rate of return of 12%, you can obtain it by selling some of your holdings of the risk-free asset and using the proceeds to buy the equally weighted market portfolio. If this is the way you decide to revise your portfolio, what will the set of weights in your portfolio be? If you hold only the risk-free asset and the market portfolio, what set of weights would give you an expected 12% return?

6.17 The market price of a security is \$40, the security's expected rate of return is 13%, the riskless rate of interest is 7%, and the market risk premium, $[E(R_m) - R_f]$, is 8%. What will be the security's current price if its expected future payoff remains the same but the covariance of its rate of return with the market portfolio doubles?

6.18 Suppose you are the manager of an investment fund in a two-parameter economy. Given the following forecast:

$$E(R_m) = .16, \qquad \sigma(R_m) = .20, \qquad R_f = .08$$

(a) Would you recommend investment in a security with $E(R_j) = .12$ and $\text{COV}(R_j, R_m) = .01$? (*Note:* Assume that this price change has no significant effect on the position of the security market line.)

(b) Suppose that in the next period security R_j has earned only 5% over the preceding period. How would you explain this ex post return?

6.19 Why is the separation principle still valid in a world with

(a) nonmarketable assets?

(b) a nonstochastic risk-free rate?

6.20 Assume that the mean-variance opportunity set is constructed from only two risky assets, A and B. Their variance-covariance matrix is

$$\sum = \begin{bmatrix} .0081 & 0 \\ 0 & .0025 \end{bmatrix}.$$

Asset A has an expected return of 30%, and Asset B has an expected return of 20%. Answer the following questions:

(a) Suppose investor I chooses his "market portfolio" to consist of 75% in asset A and 25% in asset B, whereas investor J chooses a different "market portfolio" with 50% in asset A and 50% in asset B.

Weights chosen by I are $[\,.75 \quad .25\,]$.

Weights chosen by J are $[\,.50 \quad .50\,]$.

Given these facts, what beta will each investor calculate for asset A?

(b) Given your answer to part (a), which of the following is true and why?

1. Investor I will require a higher rate of return on asset A than will investor J.
2. They will both require the same return on asset A.
3. Investor J will require a higher rate of return on asset A than will investor I.

(c) Compute the zero-beta portfolios and the equations for the security market line for each investor.

6.21 Ms. Bethel, manager of the Humongous Mutual Fund, knows that her fund currently is well diversified and that it has a CAPM beta of 1.0. The risk-free rate is 8% and the CAPM risk premium,

$\left[E(R_m) - R_f\right]$, is 6.2%. She has been learning about measures of risk and knows that there are (at least) two factors: changes in industrial production index, $\overline{\delta}_1$, and unexpected inflation, $\overline{\delta}_2$. The APT equation is

$$E(R_i) - R_f = \left[\overline{\delta}_1 - R_f\right] b_{i1} + \left[\overline{\delta}_2 - R_f\right] b_{i2},$$

$$E(R_i) = .08 + [.05]b_{i1} + [.11]b_{i2}.$$

(a) If her portfolio currently has a sensitivity to the first factor of $b_{p1} = -.5$, what is its sensitivity to unexpected inflation?

(b) If she rebalances her portfolio to keep the same expected return but reduce her exposure to inflation to zero (i.e., $b_{p2} = 0$), what will its sensitivity to the first factor become?

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